Lattice-based QCD equation of state at finite baryon density: Cluster Expansion Model

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$$\frac{p(T,\mu_B)}{T^4} = p_0(T) - \frac{2}{27\pi^2} \frac{\hat{b}_1^2}{\hat{b}_2} \left\{ 4\pi^2 \left[\text{Li}_2(x_+) + \text{Li}_2(x_-) \right] + 3\left[\text{Li}_4(x_+) + \text{Li}_4(x_-) \right] \right\}$$

Based on 1708.02852 and 1711.01261 in collaboration with A. Pasztor, Z. Fodor, S.D. Katz, J. Steinheimer, O. Philipsen, H. Stoecker

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QCD phase diagram: towards finite density



- QCD equation of state at $\mu_B = 0$ available from lattice QCD
- No direct LQCD simulations at finite μ_B but recently a lot of LQCD data which helps constrain phenomenological models

QCD thermodynamics with fugacity expansion

$$\frac{p(T,\mu_B)}{T^4} = \sum_{k=0}^{\infty} p_k(T) \cosh\left(\frac{k\,\mu_B}{T}\right) = \sum_{k=-\infty}^{\infty} \tilde{p}_{|k|}(T) \, e^{k\mu_B/T}$$

No sign problem on the lattice at imaginary $\mu_B \rightarrow i \tilde{\mu}_B$

Observables obtain trigonometric Fourier series form

Baryon density:
$$\frac{\rho_B(T, i\tilde{\mu}_B)}{T^3} = i \sum_{k=1}^{\infty} b_k(T) \sin\left(\frac{k\tilde{\mu}_B}{T}\right), \quad b_k(T) \equiv k p_k(T)$$

$$b_k(T) = \frac{2}{\pi T^4} \int_0^{\pi T} d\tilde{\mu}_B \left[\operatorname{Im} \rho_B(T, i\tilde{\mu}_B) \right] \sin(k \, \tilde{\mu}_B / T)$$

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Ideal (Boltzmann) HRG:

$$rac{
ho_B}{T^3} = b_1(T) \, \sinh\left(rac{\mu_B}{T}
ight)$$

Massless quarks (Stefan-Boltzmann limit): $b_k^{SB} = \frac{(-1)^{k+1}}{k} \frac{4 \left[3 + 4 \left(\pi k\right)^2\right]}{27 \left(\pi k\right)^2}$

Lattice QCD results on Fourier coefficients



- Consistent with HRG at low temperatures
- Consistent with approach to the Stefan-Boltzmann limit
- b_2 visibly departs from zero above $T \sim 160 \text{ MeV}$

HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) [Rischke et al., Z. Phys. C '91]



V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852

- Non-zero $b_k(T)$ for $k \ge 2$ signal deviation from ideal HRG
- EV interactions between baryons ($b \approx 1 \text{ fm}^3$) reproduce lattice trend

Higher-order coefficients from lower ones

Feature of the EV-like models: temperature-independent ratios

$$\alpha_3 = \frac{b_1(T)}{[b_2(T)]^2} b_3(T), \qquad \alpha_4 = \frac{[b_1(T)]^2}{[b_2(T)]^3} b_4(T), \qquad \dots \qquad \alpha_k = \frac{[b_1(T)]^{k-2}}{[b_2(T)]^{k-1}} b_k(T)$$

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Observation: α_3 and α_4 are *T*-independent in lattice data

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Feature of the EV-like models: temperature-independent ratios



Observation: α_3 and α_4 are *T*-independent in lattice data Ratios are consistent with Stefan-Boltzmann limit of massless quarks

Cluster Expansion Model — CEM

a model for QCD equation of state at finite baryon density

V. Vovchenko, J. Steinheimer, O. Philipsen, H. Stoecker, 1711.01261, work in progress

Cluster Expansion Model (CEM)

Model formulation:

• Fugacity expansion for baryon number density

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T)$$

- $b_1(T)$ and $b_2(T)$ are model input
- All higher order coefficients are predicted: $b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$

Physical picture: Hadron gas with repulsion at moderate *T*, "weakly" interacting quarks and gluons at high *T*

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Resummed analytic form:

$$\frac{\rho_B(T,\mu_B)}{T^3} = -\frac{2}{27\pi^2} \frac{\hat{b}_1^2}{\hat{b}_2} \left\{ 4\pi^2 \left[\text{Li}_1(x_+) - \text{Li}_1(x_-) \right] + 3 \left[\text{Li}_3(x_+) - \text{Li}_3(x_-) \right] \right\}$$
$$\hat{b}_{1,2} = \frac{b_{1,2}(T)}{b_{1,2}^{\text{SB}}}, \quad x_{\pm} = -\frac{\hat{b}_2}{\hat{b}_1} e^{\pm \mu_B/T}, \quad \text{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$
$$8/17$$

CEM: Baryon number susceptibility



Lattice data from 1112.4416 (Wuppertal-Budapest), 1701.04325 (HotQCD)

Model inputs used:

- CEM-LQCD: $b_1(T)$ and $b_2(T)$ from LQCD simulations at imaginary μ_B
- CEM-HRG: $b_1(T)$ and $b_2(T)$ from excluded-volume HRG

CEM: Higher-order susceptibilities



Lattice data on higher-order susceptibilities validate CEM

CEM: Higher-order susceptibilities

$$\chi_{k}^{B}(T,\mu_{B}) = -\frac{2}{27\pi^{2}}\frac{\hat{b}_{1}^{2}}{\hat{b}_{2}}\left\{4\pi^{2}\left[\operatorname{Li}_{2-k}(x_{+}) + (-1)^{k}\operatorname{Li}_{2-k}(x_{-})\right] + 3\left[\operatorname{Li}_{4-k}(x_{+}) + (-1)^{k}\operatorname{Li}_{4-k}(x_{-})\right]\right\}$$



To be verified by future lattice data

CEM: Higher-order susceptibilities

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Taylor expansion of the QCD pressure:

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \frac{\chi_2^B(T)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T)}{4!}(\mu_B/T)^4 + \dots$$

Radius of convergence $r_{\mu/T}$ of the expansion is the distance to the nearest singularity of p/T^4 in the complex μ_B/T plane, which could point to the QCD critical point

Lattice QCD strategy: Estimate $r_{\mu/T}$ from few leading terms [M. D'Elia et al., 1611.08285; S. Datta et al., 1612.06673; A. Bazavov et al., 1701.04325] Taylor expansion of the QCD pressure:

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CEM:
$$\chi_1^B \propto \text{Li}\left(-\frac{\hat{b}_2}{\hat{b}_1}e^{\mu_B/T}\right) \Rightarrow (\mu_B/T)_c = \pm \ln\left(\frac{\hat{b}_1}{\hat{b}_2}\right) \pm i\pi$$

Singularity in the *complex plane* \rightarrow ratio estimators *do not work*

CEM: Radius of convergence



Radius of convergence of Taylor expansion sees Roberge-Weiss transition?

- At $T > T_{RW}$ expected $\left[\frac{\mu_B}{T}\right]_c = \pm i\pi$ [Roberge, Weiss, NPB '86]
- Puts CEM in contrast to various critical point estimates

 $T_{RW} \sim 208 \text{ MeV}$ [C. Bonati et al., 1602.01426] 13/17

Extracting $b_1(T)$ and $b_2(T)$ from susceptibilities

CEM: All χ_k^B determined by b_1 and b_2 at a given temperature

Reverse prescription: Extract $b_1(T)$ and $b_2(T)$ from two independent (combinations of) χ_k^B , assuming that CEM is valid

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Example: $b_1(T)$, $b_2(T)$ from HotQCD data for χ_2^B and χ_4^B/χ_2^B at $\mu_B = 0$



14/17

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Implies accuracy of CEM and consistency between LQCD data of different groups 14/17

CEM: Observables at finite μ_B



- Non-monotonic μ_B dependence of χ_4^B/χ_2^B and χ_6^B/χ_2^B
- Ratios consistent with free Fermi gas in the limit of large μ_B
- $\chi_6^B/\chi_2^B \lesssim 0$ in the STAR-BES range

Integrating the baryon number density

$$\frac{\rho_B}{T^3} = -\frac{2}{27\pi^2} \frac{\hat{b}_1^2}{\hat{b}_2} \left\{ 4\pi^2 \left[\text{Li}_1(x_+) - \text{Li}_1(x_-) \right] + 3 \left[\text{Li}_3(x_+) - \text{Li}_3(x_-) \right] \right\}$$

one obtains the scaled pressure $p(T, \mu_B)/T^4$ in CEM

$$\frac{p(T,\mu_B)}{T^4} = p_0(T) - \frac{2}{27\pi^2} \frac{\hat{b}_1^2}{\hat{b}_2} \left\{ 4\pi^2 \left[\text{Li}_2(x_+) + \text{Li}_2(x_-) \right] + 3\left[\text{Li}_4(x_+) + \text{Li}_4(x_-) \right] \right\}$$

which provides the full equation of state within the model

Full model input:

- Fourier coefficients $b_1(T)$ and $b_2(T) \leftarrow LQCD$ at imaginary μ_B
- μ_B -independent part of pressure $p_0(T) \leftarrow LQCD$ at $\mu_B = 0$

Useful for hydro at finite baryon density

Summary

- Lattice QCD data at imaginary μ constrain phenomenological models
- Initial deviations from uncorrelated gas of hadrons can be understood in terms of repulsive baryonic interactions
- Cluster expansion model (CEM) combines hadron gas with deconfinement and is consistent with presently available lattice data, both at $\mu = 0$ and imaginary μ_B
- Radius of convergence of Taylor expansion at $\mu = 0$ is sensitive to the Roberge-Weiss transition in the complex μ_B/T plane

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- Lattice QCD data at imaginary μ constrain phenomenological models
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Thanks for your attention!

Backup slides

CEM and effective model

Recent CEM developments: [Almasi et al., 1805.04441]

Deviations from CEM ansatz when applied to effective QCD model with chiral criticality (PQM)



although model setup is not realistic compared to lattice ($T_c^{PQM} \sim 230$ MeV)

Deviations of high order Fourier coefficients or susceptibilities from CEM ansatz may signal chiral CP, if there is one

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Expected asymptotics

• At low T/densities QCD \simeq ideal hadron resonance gas

$$\frac{p^{\text{hrg}}(T,\mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2\frac{\phi_B(T)}{T^3}\cosh\left(\frac{\mu_B}{T}\right),$$

$$\phi_B(T) = \sum_{i \in B} \int dm \,\rho_i(m) \frac{d_i \, m^2 \, T}{2\pi^2} \, K_2\left(\frac{m}{T}\right),$$

$$p_0^{hrg}(T) = \frac{\phi_M(T)}{T^3}, \quad p_1^{hrg}(T) = \frac{2\,\phi_B(T)}{T^3}, \quad p_k^{\text{hrg}}(T) \equiv 0, \, k \ge 2$$

- At high T QCD \simeq ideal gas of massless quarks and gluons

$$\frac{p^{\text{\tiny SB}}(T,\mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s} \left[\frac{7\pi^2}{60} + \frac{1}{2} \left(\frac{\mu_B}{3T} \right)^2 + \frac{1}{4\pi^2} \left(\frac{\mu_B}{3T} \right)^4 \right],$$
$$p^{\text{\tiny SB}}_0 = \frac{64\pi^2}{135}, \quad p^{\text{\tiny SB}}_k = \frac{(-1)^{k+1}}{k^2} \frac{4\left[3 + 4\left(\pi k\right)^2\right]}{27\left(\pi k\right)^2}, \quad b^{\text{\tiny SB}}_k = k \, p^{\text{\tiny SB}}_k.$$

Lattice data explore intermediate, transition region 130 < T < 230 MeV

*In this study we assume that $\mu_S = \mu_Q = 0$

Radius of convergence: Structure of Taylor coefficients

Ratio estimator works when coefficients have regular asymptotic structure: they either share the same sign or they alternate in sign

Equivalently: Limiting singularity must be at a real $(\mu_B/T)^2$ value



Negative coefficients appear starting from χ_8^B

Radius of convergence: Domb-Sykes plot

Domb-Sykes plot: $1/r_n^2$ vs 1/n, linear extrapolation to 1/n = 0 yields $r_{\mu/T}$ CEM-LQCD @ T = 160 MeV



Radius of convergence: Mercer-Roberts estimator

A more involved Mercer-Roberts estimator:



Taylor expansions for p/T^4 , χ_2^B , and χ_4^B all point to the same $\lim_{n \to \infty} r_n^{-2} \simeq 0.064 \quad \Rightarrow \quad r_{\mu/T} \simeq 3.95 \text{ at } T = 160 \text{ MeV}$

1/n

16/23

0.05

CEM: Baryon number fluctuations

Baryon number susceptibilities at $\mu_B = 0$: $\chi^B_{2n}(T) \equiv \left. \frac{\partial^{2n}(p/T^4)}{\partial (\mu_B/T)^{2n}} \right|_{\mu_B=0} = \sum_{k=1}^{\infty} \, k^{2n-1} \, b_k(T) \simeq \sum_{k=1}^{k_{\text{max}}} \, k^{2n-1} \, b_k(T).$ CEM-LQCD: $b_1(T)$ and $b_2(T)$ taken from LQCD simulations at imaginary μ_B χ_2^B 0.35 CEM-LQCD 0.30 $k_{max} = 2$ 0.25 0.20 0.15 0.10 0.05 $\mu_{\rm B}$ = 0 0.00 ∟ 100 140 180 200 120 160 220 240 T [MeV]