

# Cluster expansion model for QCD baryon number fluctuations

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Based on [Phys. Lett. B 775, 71 \(2017\)](#) and [arXiv:1711.01261 \[nucl-th\]](#)

**Thanks to:** Z. Fodor, S.D. Katz, A. Pásztor, O. Philipsen, J. Steinheimer, H. Stoecker

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# Outline

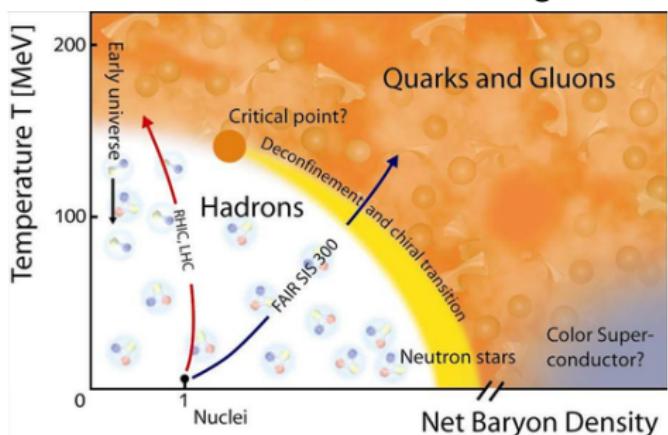
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1. Motivation: QCD equation of state at finite  $\mu_B$
2. Lattice QCD data at imaginary  $\mu_B$ 
  - Fourier coefficients of fugacity expansion
  - Interpretation in terms of baryonic excluded volume
3. Cluster expansion model
  - Fourier coefficients
  - Taylor expansion coefficients (baryon susceptibilities)
  - Radius of convergence
4. Summary and outlook

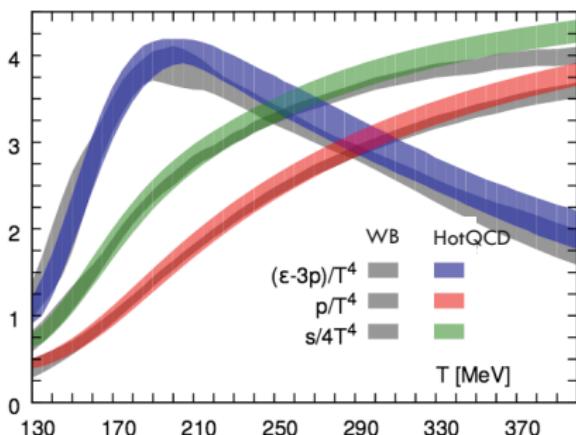
# Strongly interacting matter

- Theory of strong interactions: **Quantum Chromodynamics** (QCD)
- Basic degrees of freedom: quarks and gluons
- At smaller energies confined into hadrons: baryons ( $qqq$ ) and mesons ( $q\bar{q}$ )

Sketch of the QCD Phase Diagram



Lattice QCD EoS at  $\mu_B = 0$



Relevant for heavy-ion collisions, neutron stars, early universe

First-principle tool: **Lattice QCD**. Direct simulations restricted to  $\mu_B = 0$ .

What can we learn about EoS at finite  $\mu_B$  from lattice and effective models?

## Lattice-based methods for equation of state at finite $\mu_B$

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- Taylor expansion [Allton et al.; Gavai, Gupta; HotQCD Collaboration]

$$\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \frac{\chi_2^B(T, 0)}{2!} (\mu_B/T)^2 + \frac{\chi_4^B(T, 0)}{4!} (\mu_B/T)^4 + \dots$$

$\chi_k^B$  – cumulants (susceptibilities) of net baryon distribution

Can be computed in Lattice QCD at  $\mu_B = 0$

- Analytic continuation from imaginary  $\mu_B$  [de Forcrand, Philipsen; D'Elia, Lombardo]  
No sign problem at  $\mu_B = i\tilde{\mu}_B$ : Observables can be computed at  $\mu_B^2 < 0$   
Then analytically continued to  $\mu_B^2 > 0$
- Other methods: Reweighting, complex Langevin, etc.

## QCD observables at imaginary $\mu_B$

QCD thermodynamics with **relativistic fugacity expansion**:

Pressure: 
$$\frac{p(T, \mu_B)}{T^4} = \sum_{k=0}^{\infty} p_k(T) \cosh\left(\frac{k \mu_B}{T}\right),$$

Net baryon density: 
$$\frac{\rho_B(T, \mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T) \sinh\left(\frac{k \mu_B}{T}\right), \quad b_k(T) \equiv k p_k(T)$$

Lattice QCD is problematic at real  $\mu$  but tractable at **imaginary  $\mu$**

$\mu_B \rightarrow i \tilde{\mu}_B \Rightarrow$  QCD observables obtain **trigonometric Fourier series** form

Pressure: 
$$\frac{p(T, i \tilde{\mu}_B)}{T^4} = \sum_{k=0}^{\infty} p_k(T) \cos\left(\frac{k \tilde{\mu}_B}{T}\right),$$

Net baryon density: 
$$\frac{\rho_B(T, i \tilde{\mu}_B)}{T^3} = i \sum_{k=1}^{\infty} b_k(T) \sin\left(\frac{k \tilde{\mu}_B}{T}\right), \quad b_k(T) \equiv k p_k(T)$$

Coefficients  $b_k(T)$  can and are now being calculated in LQCD

## Expected asymptotics

- At low T/densities  $\text{QCD} \simeq \text{ideal hadron resonance gas}$

$$\frac{p^{\text{hrg}}(T, \mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2 \frac{\phi_B(T)}{T^3} \cosh\left(\frac{\mu_B}{T}\right),$$

$$\phi_B(T) = \sum_{i \in B} \int dm \rho_i(m) \frac{d_i m^2 T}{2\pi^2} K_2\left(\frac{m}{T}\right),$$

$$p_0^{\text{hrg}}(T) = \frac{\phi_M(T)}{T^3}, \quad p_1^{\text{hrg}}(T) = \frac{2\phi_B(T)}{T^3}, \quad p_k^{\text{hrg}}(T) \equiv 0, \quad k = 2, 3, \dots$$

- At high T  $\text{QCD} \simeq \text{ideal gas of massless quarks and gluons}$

$$\frac{p^{\text{SB}}(T, \mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s} \left[ \frac{7\pi^2}{60} + \frac{1}{2} \left( \frac{\mu_f}{T} \right)^2 + \frac{1}{4\pi^2} \left( \frac{\mu_f}{T} \right)^4 \right], \quad \mu_f = \frac{\mu_B}{3} *,$$

$$p_0^{\text{SB}} = \frac{64\pi^2}{135}, \quad p_k^{\text{SB}} = \frac{(-1)^{k+1}}{k^2} \frac{4[3 + 4(\pi k)^2]}{27(\pi k)^2}, \quad b_k^{\text{SB}} = k p_k^{\text{SB}}.$$

This work explores intermediate, transition region  $130 < T < 230$  MeV

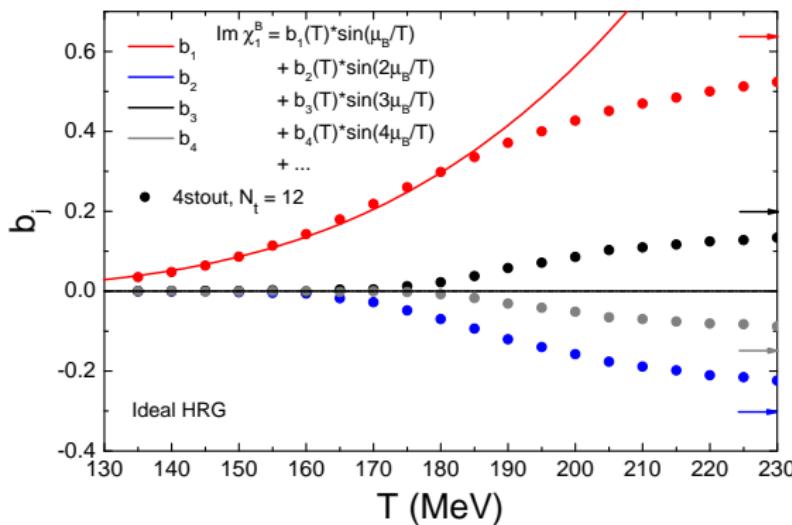
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\*In this study we assume that  $\mu_S = \mu_Q = 0$

# Lattice QCD results on imaginary $\mu_B$ observables

Coefficients  $b_k(T)$  of net-baryon expansion are now calculated on the lattice

$$\frac{\rho_B(T, i\tilde{\mu}_B)}{T^3} = i \sum_{j=1}^{\infty} b_j(T) \sin(j\tilde{\mu}_B/T)$$

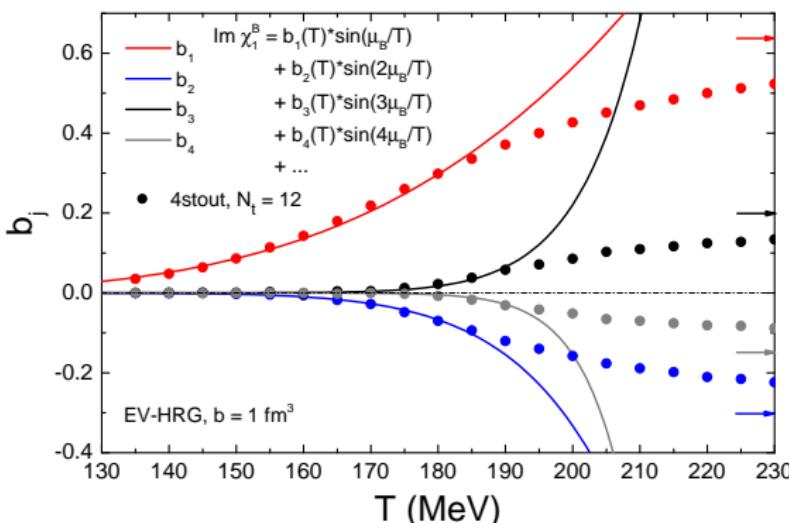


- Ideal HRG describes well  $b_1(T)$  at small temperatures
- All four coefficients appear to converge slowly to Stefan-Boltzmann limit
- What is the mechanism of appearance of non-zero  $b_k$  for  $k > 1$ ?

# Imaginary $\mu_B$ and repulsive baryonic interactions

Repulsive baryonic interactions with excluded-volume [Rischke et al., Z. Phys. C '91]

$$V \rightarrow V - bN \quad \Rightarrow \quad p_B(T, \mu_B) = p_B^{\text{id}}(T, \mu_B - bp)$$



HRG with baryonic EV repulsion:

$$b_1^{\text{ev}}(T) = 2 \frac{\phi_B(T)}{T^3}$$

$$b_2^{\text{ev}}(T) = -4 [b \phi_B(T)] \frac{\phi_B(T)}{T^3}$$

$$b_3^{\text{ev}}(T) = 9 [b \phi_B(T)]^2 \frac{\phi_B(T)}{T^3}$$

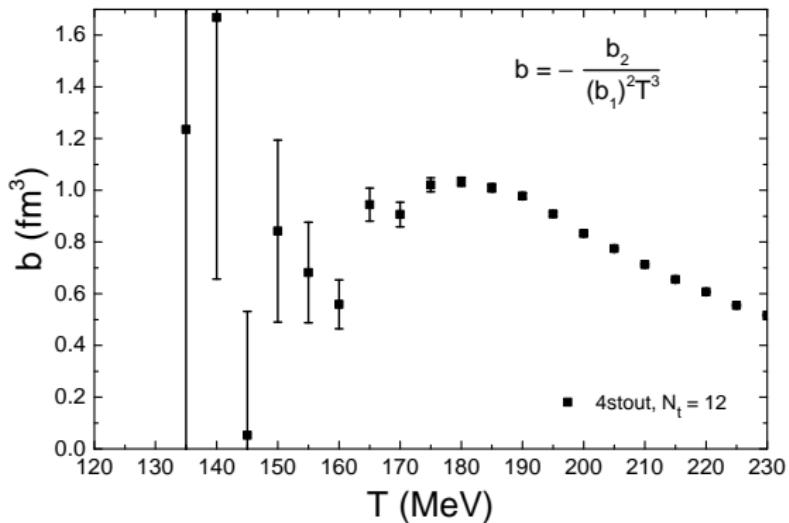
$$b_4^{\text{ev}}(T) = -\frac{64}{3} [b \phi_B(T)]^3 \frac{\phi_B(T)}{T^3}$$

- Ideal HRG describes well  $b_1(T)$  at small temperatures
- Non-zero  $b_j(T)$  for  $j \geq 2$  signal deviations from ideal HRG
- Addition of EV interactions between baryons reproduces lattice trend

## “Excluded volume” parameter from imaginary $\mu_B$ data

“Excluded volume” parameter of  $BB$  interactions can be estimated from lattice

$$b(T) = -\frac{b_2(T)}{[b_1(T)]^2 T^3}$$



$b(T)$  mostly consistent with  $1 \text{ fm}^3$  at  $T < 190 \text{ MeV}$

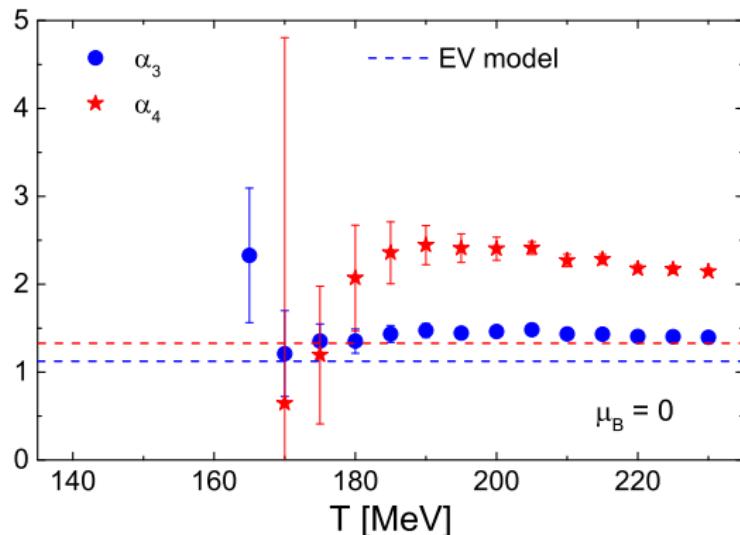
## Relation between leading and higher order coefficients

EV-HRG describes similarly well leading four coefficients

A particular feature of the model: **temperature-independent** ratios

$$\alpha_3 = \frac{b_1(T)}{[b_2(T)]^2} b_3(T), \quad \alpha_4 = \frac{[b_1(T)]^2}{[b_2(T)]^3} b_4(T), \quad \dots$$

Also hold true for many other models with short-range interaction



Excluded volume model:

$$\alpha_3^{EV} = 1.125, \quad \alpha_4^{EV} = 1.333$$

$\alpha_3$  and  $\alpha_4$  are approximately **T-independent** on the lattice, EV somewhat off

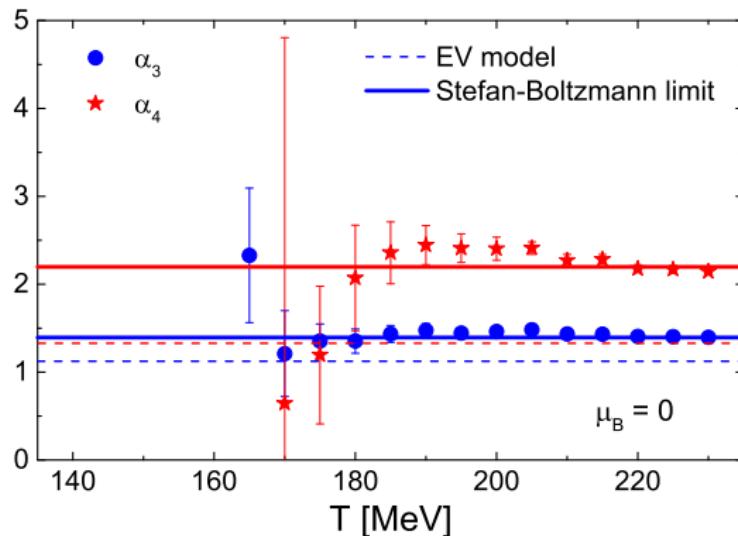
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Excluded volume model:

$$\alpha_3^{EV} = 1.125, \quad \alpha_4^{EV} = 1.333$$

Stefan-Boltzmann limit:

$$\alpha_3^{SB} \simeq 1.394, \quad \alpha_4^{SB} \simeq 2.198$$

$\alpha_3$  and  $\alpha_4$  are approximately **T-independent** on the lattice, EV somewhat off

Ratios are consistent with the **Stefan-Boltzmann limit** of massless quarks

## Cluster Expansion Model (CEM)

$\alpha_3$  and  $\alpha_4$  are consistent with SB limit. Now assume the same for all higher-order coefficients

### CEM formulation:

- $b_1(T)$  and  $b_2(T)$  are model input
- All higher order coefficients are then predicted

$$b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$$

- All observables are calculated from fugacity expansion for baryon density

$$\frac{\rho_B(T)}{T^3} = \chi_1^B(T) = \sum_{k=1}^{\infty} b_k(T) \sinh(k \mu_B / T)$$

Fugacity expansion convergence criterion is given by the ratio test:

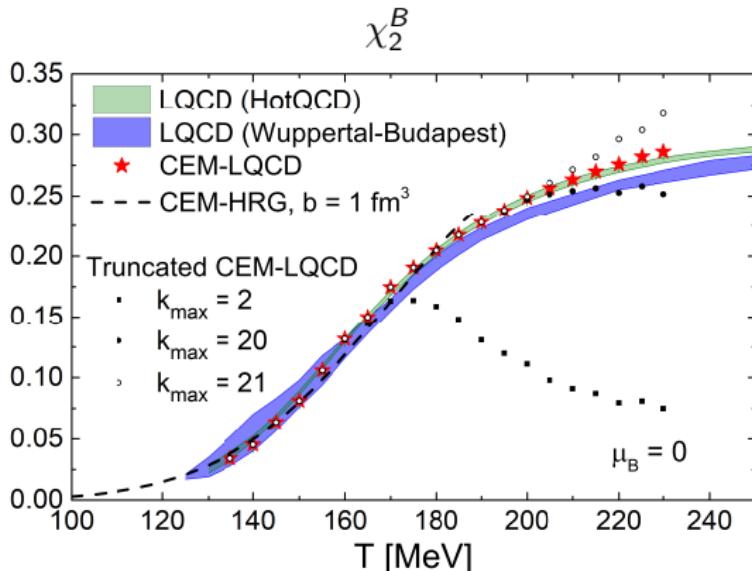
$$\lim_{k \rightarrow \infty} \left| \frac{b_{k+1}(T) \sinh \left[ \frac{(k+1) \mu_B}{T} \right]}{b_k(T) \sinh \left[ \frac{k \mu_B}{T} \right]} \right| = \left| \frac{b_2(T) b_1^{SB}}{b_1(T) b_2^{SB}} \right| e^{\frac{|\mu_B|}{T}} < 1.$$

# CEM: Baryon number fluctuations

Baryon number susceptibilities at  $\mu_B = 0$ :

$$\chi_{2n}^B(T) \equiv \frac{\partial^{2n}(p/T^4)}{\partial(\mu_B/T)^{2n}} \Big|_{\mu_B=0} = \sum_{k=1}^{\infty} k^{2n-1} b_k(T).$$

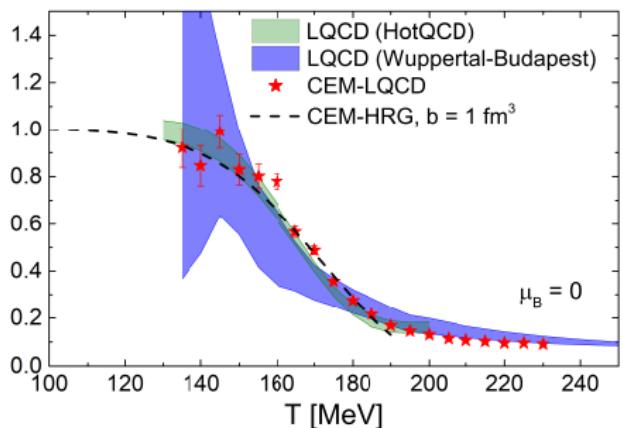
- CEM-LQCD:  $b_1(T)$  and  $b_2(T)$  taken from LQCD simulations at imaginary  $\mu_B$   
CEM-HRG:  $b_1(T)$  and  $b_2(T)$  from EV-HRG model with  $b = 1 \text{ fm}^3$



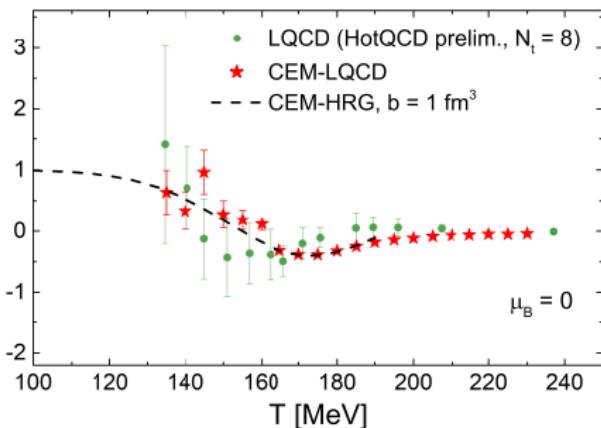
# CEM: 4th and 6th order ratios

$$\chi_{2n}^B(T) \equiv \left. \frac{\partial^{2n}(p/T^4)}{\partial(\mu_B/T)^{2n}} \right|_{\mu_B=0} = \sum_{k=1}^{\infty} k^{2n-1} b_k(T).$$

$\chi_4^B/\chi_2^B$



$\chi_6^B/\chi_2^B$



Consistency with available LQCD data

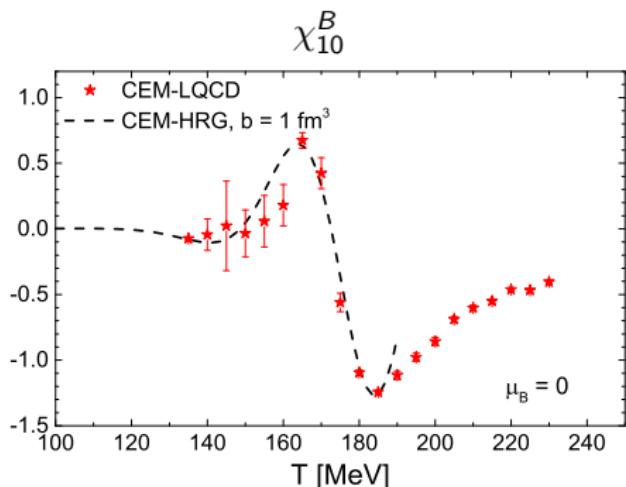
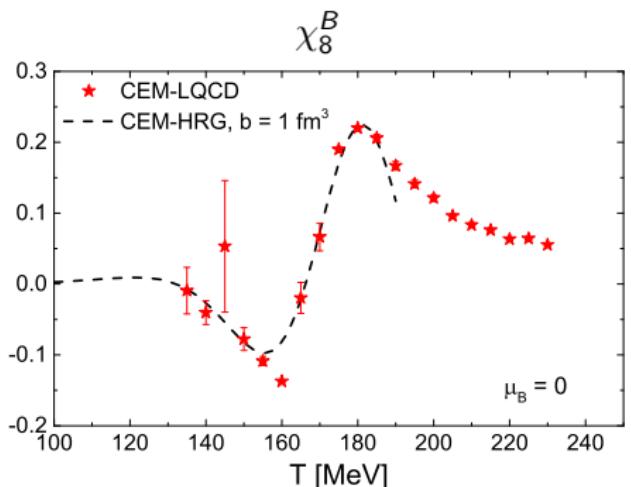
Hadronic description with interactions (CEM-HRG) works up to  $T \simeq 185$  MeV

V.V., J. Steinheimer, O. Philipsen, H. Stoecker, 1711.01261

LQCD data from 1507.04627 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)

# CEM: predictions for high orders

$$\chi_{2n}^B(T) \equiv \left. \frac{\partial^{2n}(p/T^4)}{\partial(\mu_B/T)^{2n}} \right|_{\mu_B=0} = \sum_{k=1}^{\infty} k^{2n-1} b_k(T).$$



To be verified on the lattice

## Radius of convergence

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Taylor expansion of QCD pressure:

$$\frac{p(T, \mu_B)}{T^4} = \frac{p(T, 0)}{T^4} + \frac{\chi_2^B(T)}{2!} (\mu_B/T)^2 + \frac{\chi_4^B(T)}{4!} (\mu_B/T)^4 + \dots$$

Radius of convergence  $r_{\mu/T}$  of the expansion is the distance to the nearest singularity of  $p/T^4$  in the **complex**  $\mu_B/T$  plane at a given temperature  $T$

If the nearest singularity is at a real  $\mu_B/T$  value, this could point to the **QCD critical point**

Lattice QCD strategy: Estimate  $r_{\mu/T}$  from few leading terms

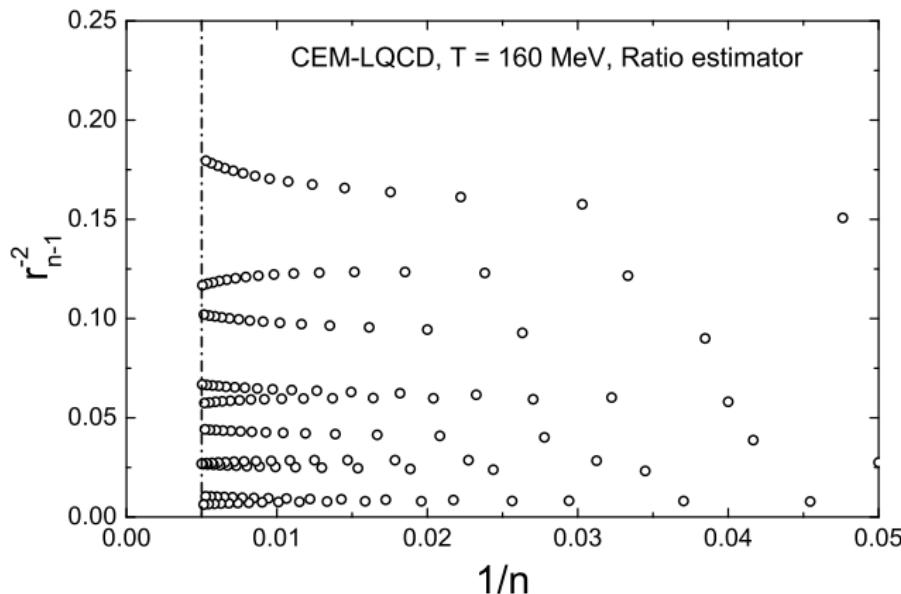
M. D'Elia et al., 1611.08285; S. Datta et al., 1612.06673; A. Bazavov et al., 1701.04325

Ratio estimator:  $r_n = \left| \frac{(2n+2)(2n+1)\chi_{2n}^B}{\chi_{2n+2}^B} \right|^{1/2}, \quad r_{\mu/T} = \lim_{n \rightarrow \infty} r_n$

CEM allows to analyze  $r_n$  to very high order

## Radius of convergence: Domb-Sykes plot

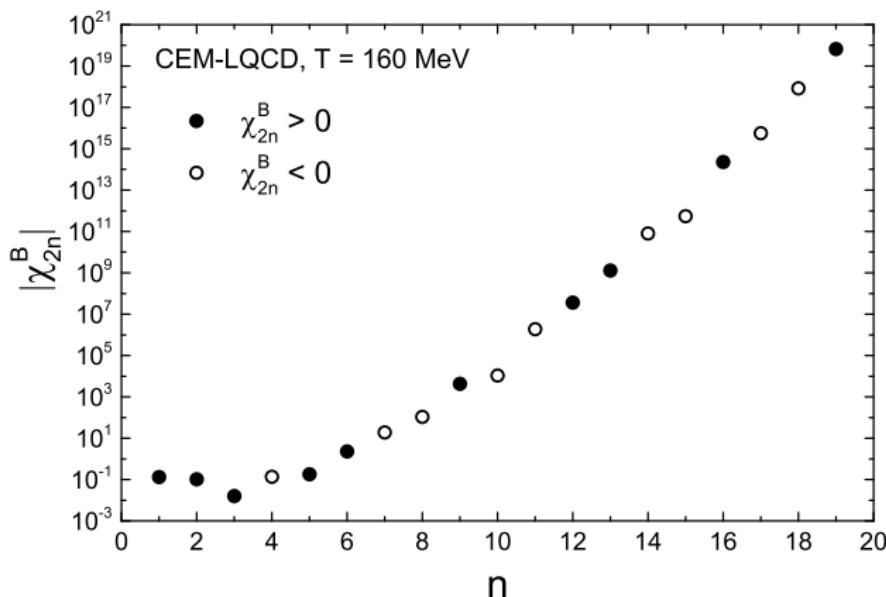
Domb-Sykes plot:  $1/r_n^2$  vs  $1/n$ , linear extrapolation to  $1/n = 0$  yields  $r_{\mu/T}$   
CEM-LQCD @  $T = 160$  MeV



$\lim_{n \rightarrow \infty} r_n$  does not exist!

## Radius of convergence: Structure of Taylor coefficients

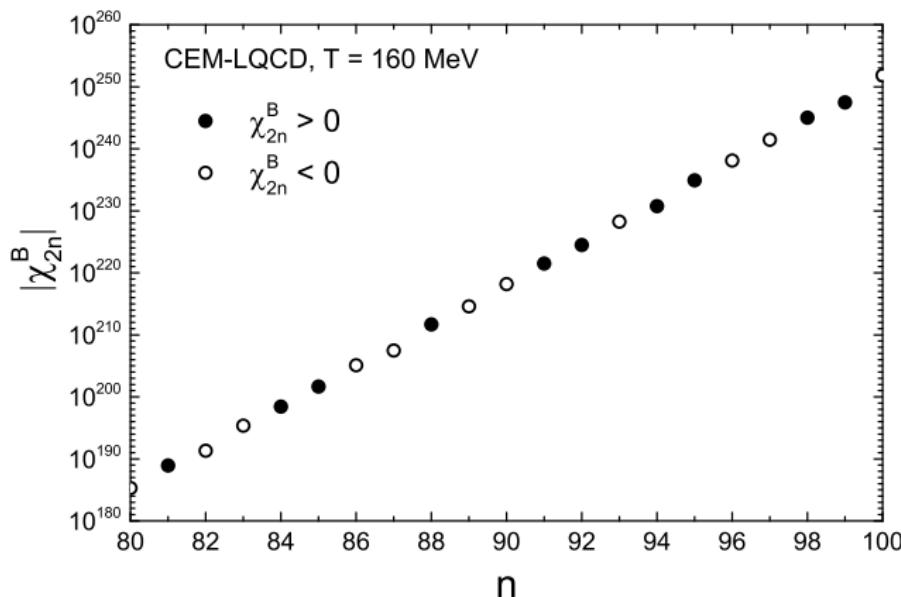
Ratio estimator works only when coefficients have **regular asymptotic structure**:  
they either share the **same sign** or they **alternate in sign**



Negative coefficients appear from  $\chi_8^B$  on

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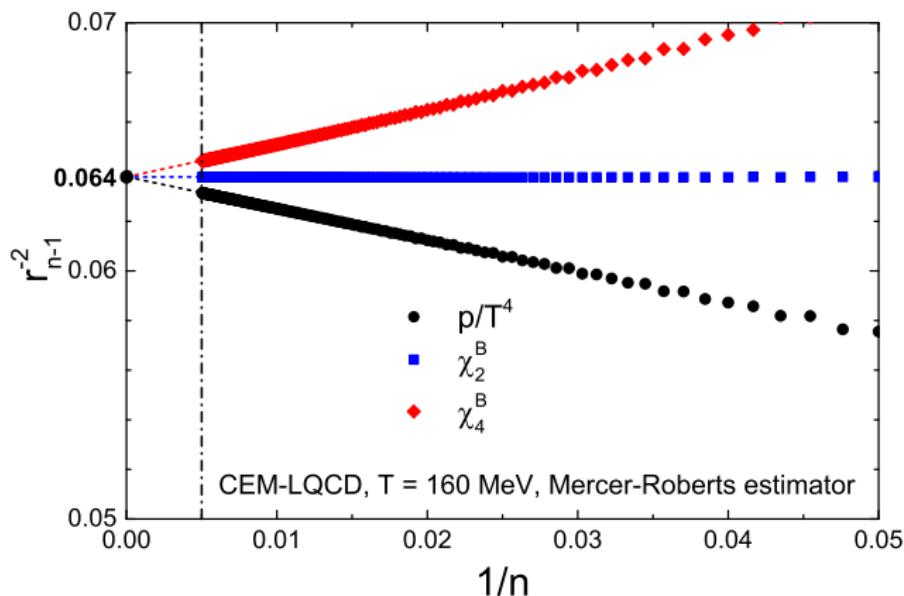
Negative coefficients appear from  $\chi_8^B$  on  
They never settle into a regular pattern

This means that limiting singularity lies in the **complex  $\mu_B/T$  plane**

## Radius of convergence: Mercer-Roberts estimator

A more involved Mercer-Roberts estimator:

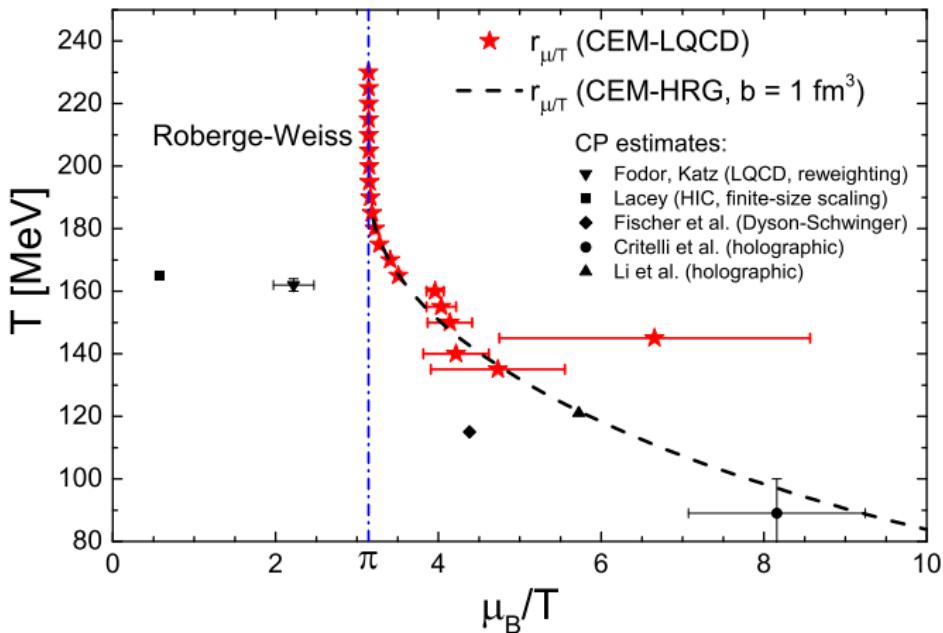
$$r_n = \left| \frac{c_{n+1} c_{n-1} - c_n^2}{c_{n+2} c_n - c_{n+1}^2} \right|^{1/4}, \quad c_n = \frac{\chi_{2n}^B}{(2n)!}.$$



Taylor expansions for  $p/T^4$ ,  $\chi_2^B$ , and  $\chi_4^B$  all point to the same  
 $\lim_{n \rightarrow \infty} r_n^{-2} \simeq 0.064 \Rightarrow r_{\mu/T} \simeq 3.95$  at  $T = 160$  MeV

# Radius of convergence: Temperature dependence

Applying the same procedure at other temperatures



Radius of convergence of Taylor expansion sees Roberge-Weiss transition?

R-W transition expected at  $T > T_{RW}$  and  $\text{Im}[\mu_B/T] = \pi$  [Roberge, Weiss, NPB '86]

Lattice estimate:  $T_{RW} \sim 200$  MeV [C. Bonati et al., 1602.01426 ]

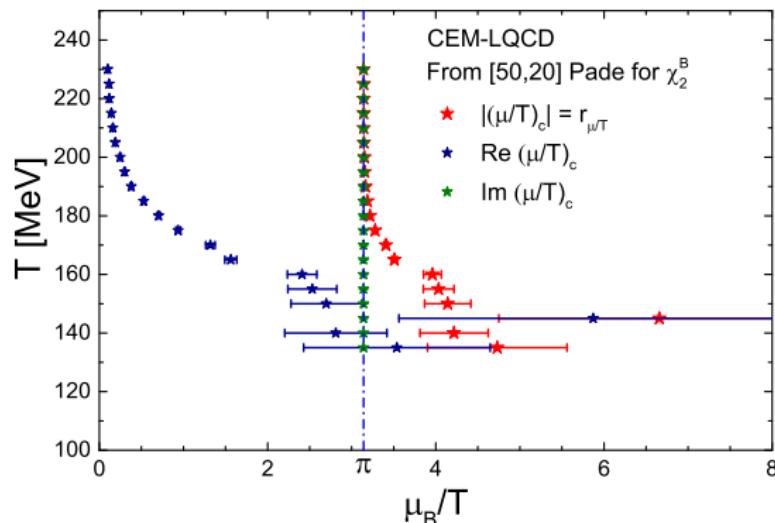
## Radius of convergence: Cross-check with Padé approximants

Padé approximant for  $\chi_2^B$ :

$$\chi_2^B(T, \mu_B/T) \approx \frac{\sum_{j=0}^m a_j (\mu_B/T)^j}{1 + \sum_{k=1}^n b_k (\mu_B/T)^k}$$

$a_j$  and  $b_k$  constructed from  $\chi_{2n}^B$  to match Taylor expansion

Poles of Padé approximants often point to true singularities of the function

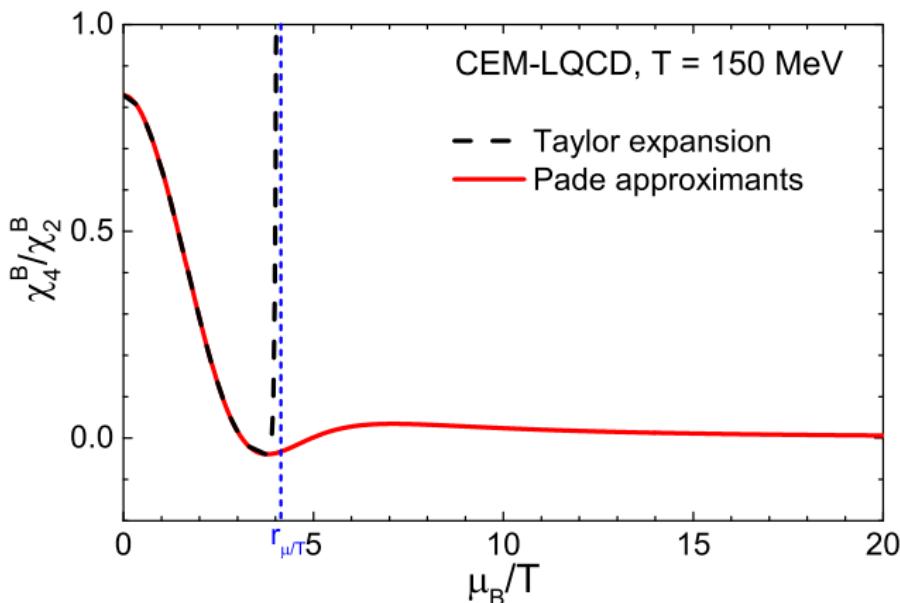


$\text{Im}[\mu_B/T]_c = \pi$ , while  $\text{Re}[\mu_B/T]_c$  decreases towards zero with temperature

## Going beyond the radius of convergence

Padé approximants allow to go beyond the radius for convergence

Example:  $\chi_4^B / \chi_2^B$  at finite  $\mu_B / T$



## Summary

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- LQCD data at imaginary  $\mu_B$  suggests presence of repulsive baryonic interactions with 2nd virial coefficient  $b \sim 1 \text{ fm}^3$  in the crossover region
- It provides a first-principle evidence for the baryonic “excluded-volume”
- CEM describes all available lattice data on net baryon susceptibilities
- Radius of convergence of Taylor expansion sees a Roberge-Weiss like transition
- No evidence for QCD phase transition at  $\mu_B/T < \pi$

## Outlook

- QCD equation of state at finite  $\mu_B/T$  within CEM
- Isospin and strangeness chemical potentials

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**Thanks for your attention!**

Backup slides

## Baryonic excluded volume

Baryon-baryon interactions seem to exhibit a repulsive core – **excluded volume**

EV model: a simple approach for repulsive interactions [Rischke et al., Z. Phys. C '91]

$$V \rightarrow V - bN \quad \Rightarrow \quad p(T, \mu) = p^{\text{id}}(T, \mu - bp)$$

### EV-HRG model

- Identical EV interactions for all baryon-baryon and antibaryon-antibaryon pairs
- Baryon-antibaryon, meson-meson, meson-baryon EV terms **neglected**
- A single parameter  $b$  characterizing interactions

Three independent subsystems: **mesons + baryons + antibaryons**

$$p(T, \mu) = p_M(T, \mu) + p_B(T, \mu) + p_{\bar{B}}(T, \mu),$$

$$p_M(T, \mu) = \sum_{j \in M} p_j^{\text{id}}(T, \mu_j) \quad \text{and} \quad p_B(T, \mu) = \sum_{j \in B} p_j^{\text{id}}(T, \mu_j - b p_B)$$

$$\text{Total density of baryons: } n_B^{\text{ev}} = (1 - b n_B^{\text{ev}}) e^{\mu_B/T} \phi_B(T) \exp\left(-\frac{b n_B^{\text{ev}}}{1 - b n_B^{\text{ev}}}\right).$$

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V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)

L. Satarov, V.V., P. Alba, M. Gorenstein, H. Stoecker, Phys. Rev. C 95, 024902 (2017)