Towards the QCD equation of state at finite density

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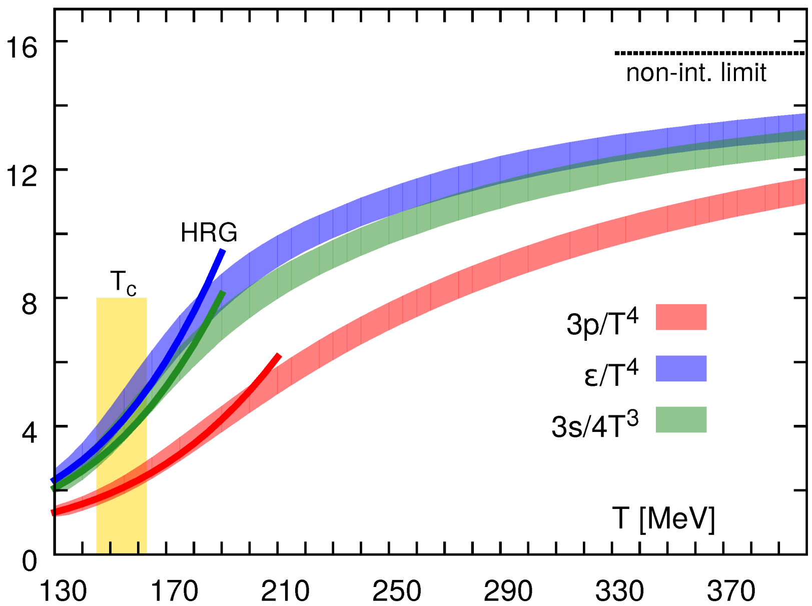
LBNL Nuclear Theory Seminar, Berkeley, USA

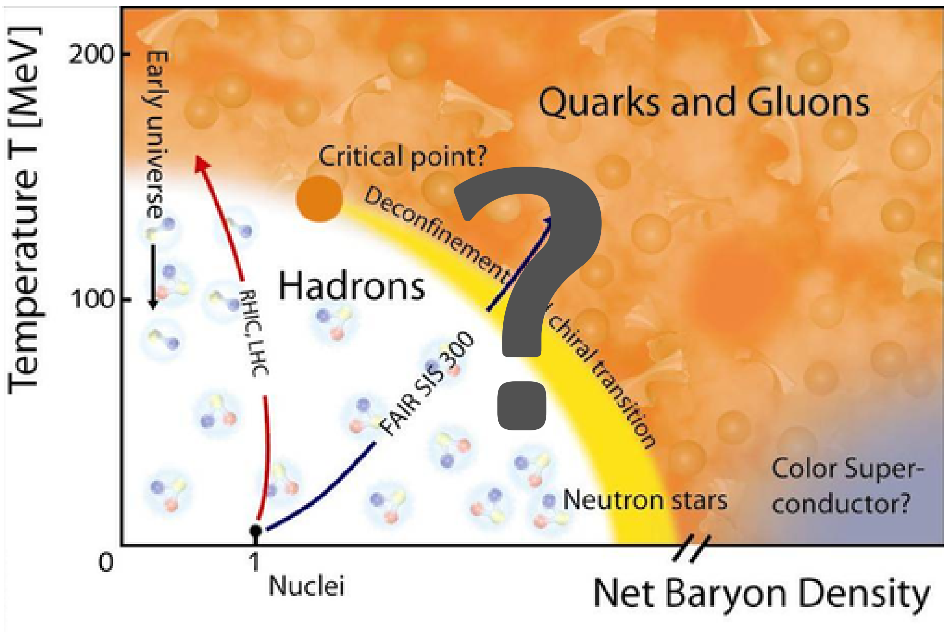
May 2, 2019

QCD phase diagram: towards finite density

?

$\mu_B = 0$ → $T - \mu_B$ plane





- QCD EoS at $\mu_B = 0$ available from lattice QCD
- QCD EoS at finite density necessary for many applications, including hydro modeling of heavy-ion collisions at RHIC, SPS, FAIR energies
- Implementation of the QCD critical point necessary to look for its signatures

Outline

1. Taylor expansion from lattice QCD
   - Model-independent method with a limited scope (small μ_B/T)
   - State-of-the-art and estimates for radius of convergence

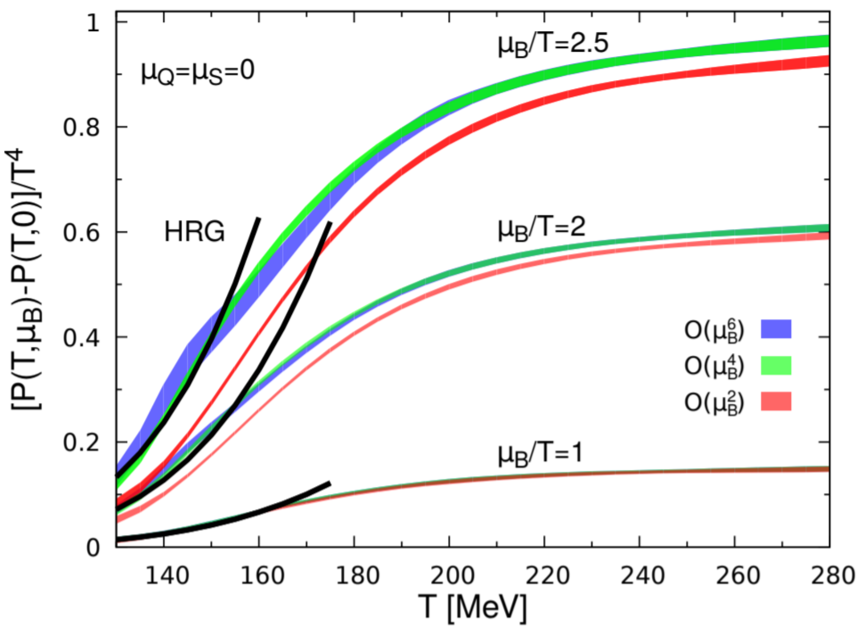
2. Lattice-based effective models
   - Cluster expansion model (CEM)
   - Hagedorn bag-like model

3. Signatures of the critical point at finite density
   - Exponential suppression of Fourier coefficients
   - Extracting the location of singularities from the lattice data

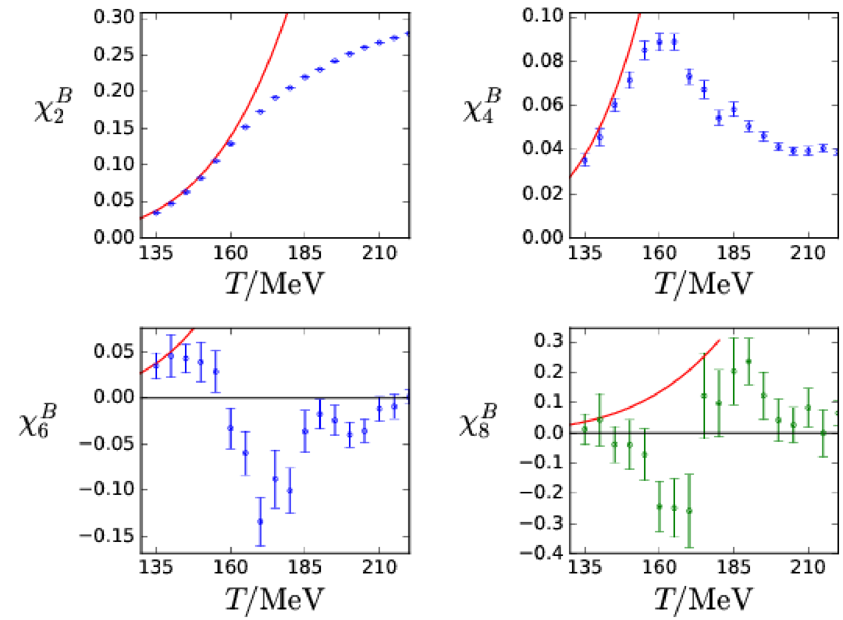
Finite μ_B EoS from Taylor expansion

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \frac{\chi_2^B(T,0)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T,0)}{4!}(\mu_B/T)^4 + \ldots$$

χ_k^B – cumulants of net baryon distribution, computed up to χ_8^B



[HotQCD collaboration, 1701.04325]



[Wuppertal-Budapest collaboration, 1805.04445]

- Off-diagonal susceptibilities also available → incorporate conservation laws $n_S = 0,\, n_Q/n_B = 0.4$
- Method inherently limited to "small" μ_B/T, within convergence radius

Taylor expansion and radius of convergence

A truncated Taylor expansion only useful within the radius of convergence. Its value is a priori unknown. Any singularity in *complex* μ_B plane will limit the convergence, it does not have to be a phase transition or a critical point

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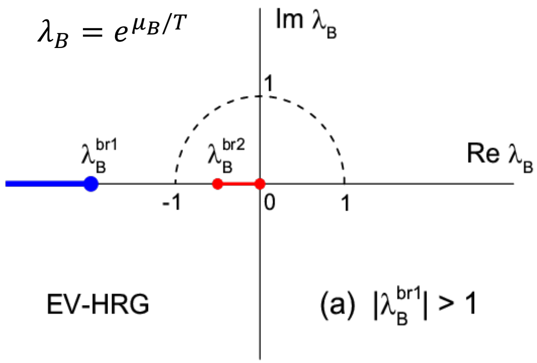
An example: HRG model with a baryonic excluded volume (EV) $V \to V - bN$



$$p(T, \mu_B) \sim W\left[b\,\phi_B(T)\,e^{\mu_B/T}\right]$$

$b \simeq 1$ fm^3

Constrained to LQCD data

[**V.V.** et al., 1708.02852]

Lambert W(z) function has a branch cut singularity at $z = -e^{-1}$, corresponds to a negative fugacity

[Taradiy, Motornenko, **V.V.**, Gorenstein, Stoecker, 1904.08259]

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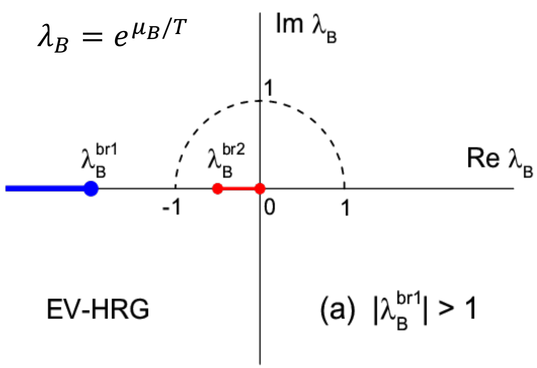

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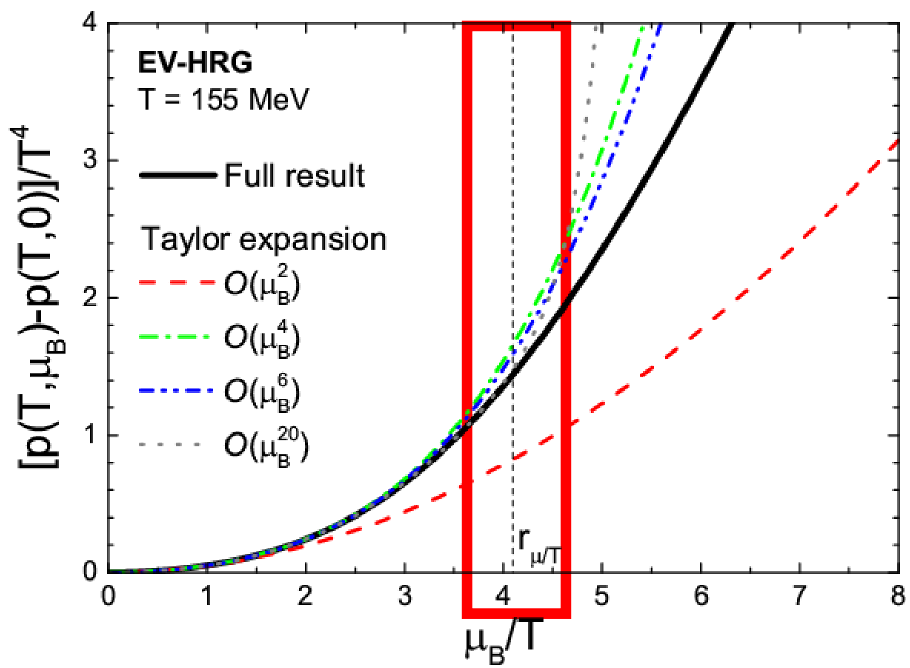
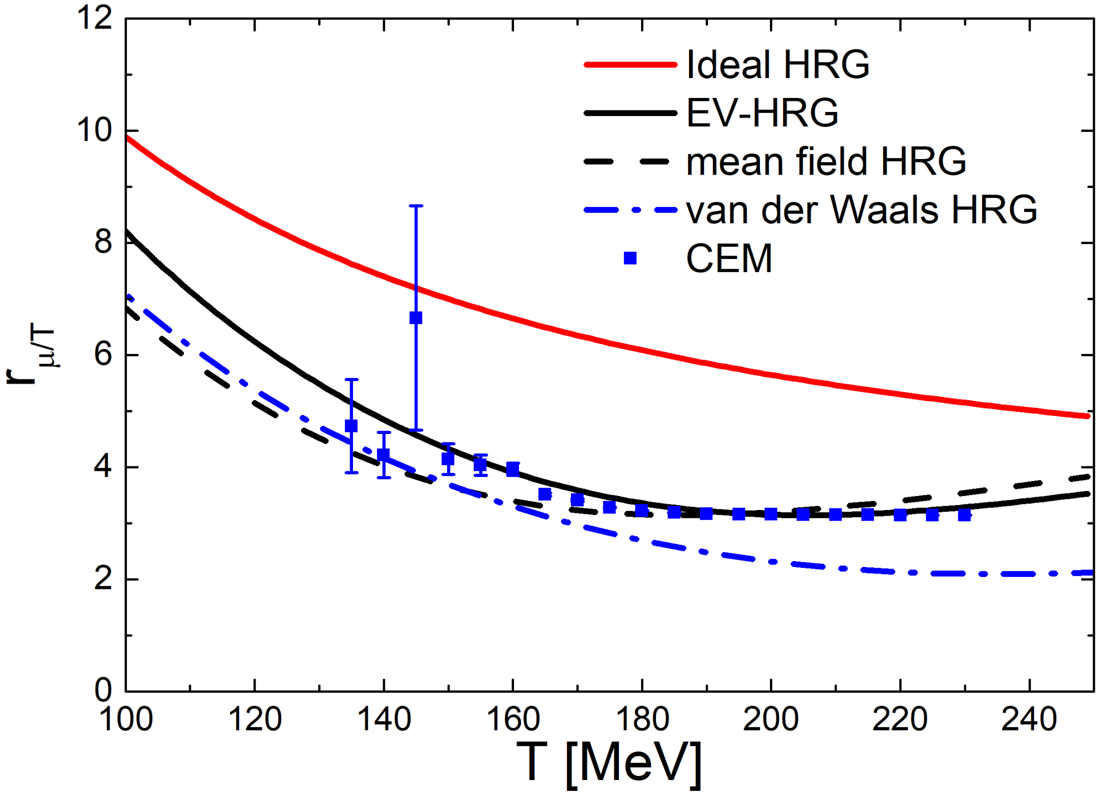


Radius of convergence from different models



Ideal HRG

Singularity in the nucleon Fermi-Dirac function

$$\left[\exp\left(\frac{\sqrt{m^2+p^2}-\mu_B}{T}\right)+1\right]^{-1}$$

EV-HRG & mean-field HRG

[**V.V.**+, 1708.02852] [Huovinen, Petreczky, 1708.02852]

Repulsive baryonic interactions. Singularity of the Lambert W function

van der Waals HRG

[**V.V.**, Gorenstein, Stoecker, 1609.03975]

Crossover singularities connected to the nuclear matter critical point at $T \sim 20$ MeV and $\mu_B \sim 900$ MeV

see also M. Stephanov, hep-lat/0603014

Cluster Expansion Model (CEM)

[**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]

Roberge-Weiss like transition: $Im\, \frac{\mu_B}{T} = \pi$

Taylor expansion likely divergent at $\mu_B/T \geq$ 3-5, regardless of existence of the QCD critical point

Recent Taylor-based EoS parameterizations

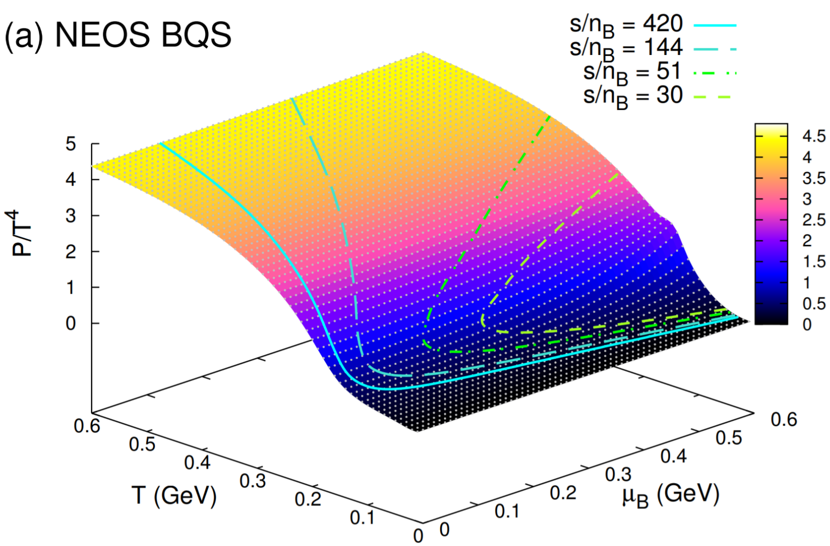
Truncated LQCD Taylor expansion

$$\frac{p}{T^4} = \sum_{i,j,k} \frac{\chi^{BQS}_{i,j,k}(T)}{i!\,j!\,k!} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

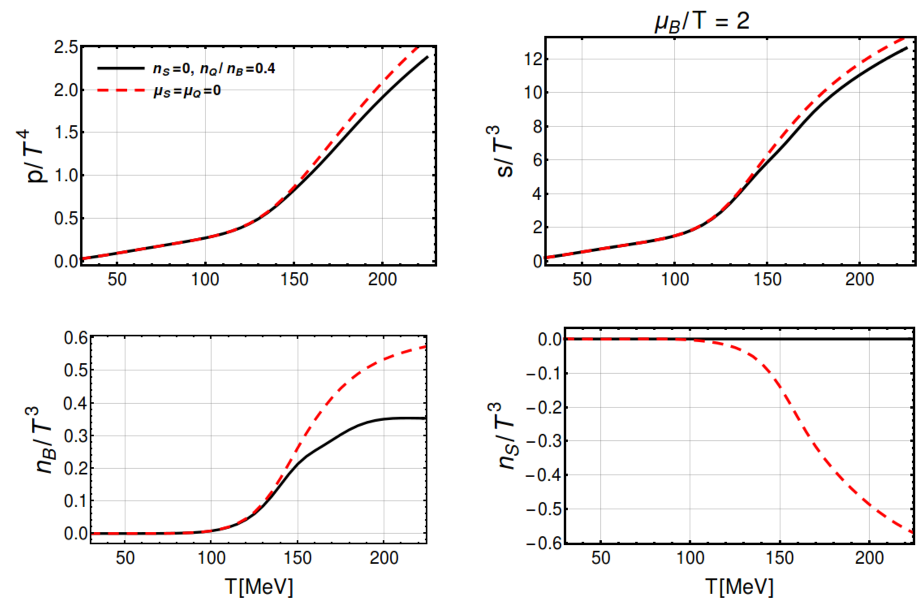
+

HRG model at smaller temperatures

$$\frac{p}{T^4} = \sum_{i\in \mathrm{hrg}} T\,\phi_i^{\mathrm{id}}(T)\, e^{b_i\mu_B/T}\, e^{q_i\mu_Q/T}\, e^{s_i\mu_S/T}$$



[Monnai, Schenke, Shen, 1902.05095]



[Noronha-Hostler, Parotto, Ratti, Stafford, 1902.06723]

- Includes the three conserved charges and conservation laws, no criticality
- Probably best one can do with Taylor expansion. Applications: RHIC BES

Truncated Taylor expansion and imaginary μ_B

Are we using all information available from lattice? Consider relativistic virial expansion (Laurent series in fugacity) and imaginary μ_B

$$\left.\frac{\rho_B}{T^3}\right|_{\mu_B=i\theta_B T} = i\sum_{k=1}^{\infty} b_k(T)\sin(k\theta_B) \qquad \Rightarrow \qquad b_k(T) = -\frac{2i}{\pi}\int_0^{\pi}\frac{\rho_B(T,i\theta_B T)}{T^3}\sin(k\,\theta_B)\,d\theta_B$$

Relativistic virial/cluster expansion — *Fourier coefficients*

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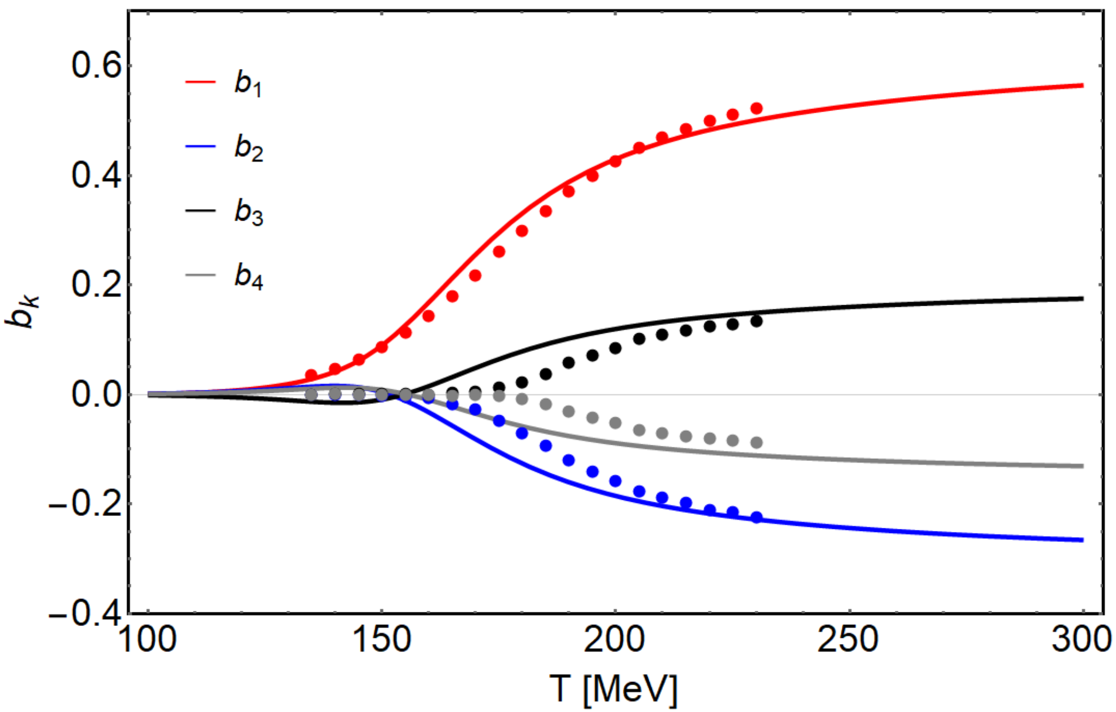
Relativistic virial/cluster expansion — *Fourier coefficients*



Lines: *Taylor expansion up to χ_B^4 using lattice data, as in 1902.06723*

Symbols: *Lattice data for b_k from imaginary μ_B*

[**V.V.**, Pasztor, Fodor, Katz, Stoecker, 1708.02852]

Quite some room for improvement at $T<200$ MeV

Cluster Expansion Model — CEM

a model for QCD equation of state at finite baryon density
constrained to both susceptibilities and Fourier coefficients

V.V., *J. Steinheimer, O. Philipsen, H. Stoecker, Phys. Rev. D 97, 114030 (2018)*
V.V. *et al., Nucl. Phys. A 982, 859 (2019)*

Cluster Expansion Model (CEM)

Model formulation:

- Cluster expansion for baryon number density

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T)$$

- $b_1(T)$ and $b_2(T)$ are model input from lattice QCD
- All higher order coefficients are predicted: $b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$

Physical picture: Hadron gas with repulsion at moderate T, QGP-like phase at high T

Summed analytic form:

$$\frac{\rho_B(T,\mu_B)}{T^3} = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_1(x_+) - \mathrm{Li}_1(x_-)\right] + 3\left[\mathrm{Li}_3(x_+) - \mathrm{Li}_3(x_-)\right]\right\}$$

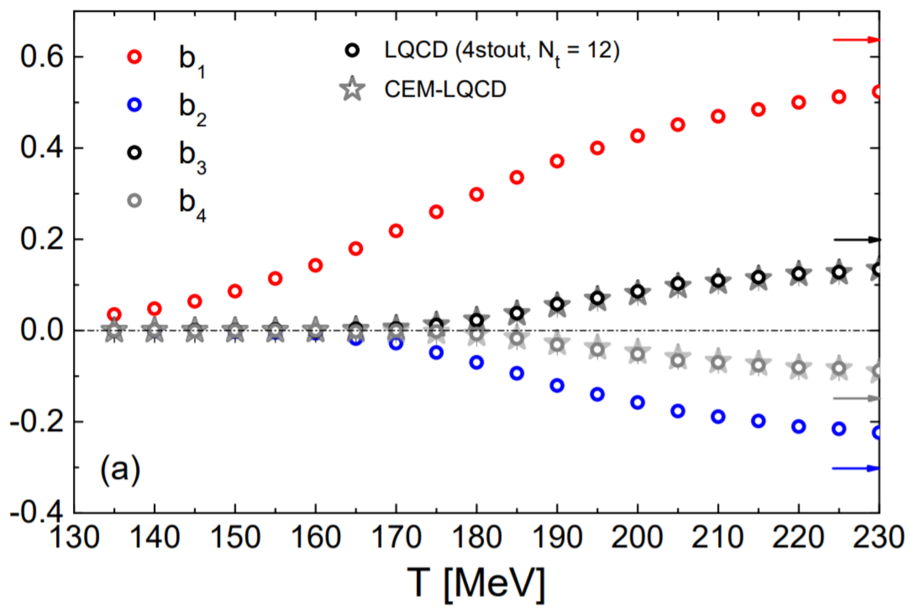
$$\hat{b}_k = \frac{b_k(T)}{b_k^{\mathrm{SB}}}, \quad x_\pm = -\frac{\hat{b}_2}{\hat{b}_1} e^{\pm\mu_B/T}, \quad \mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$$

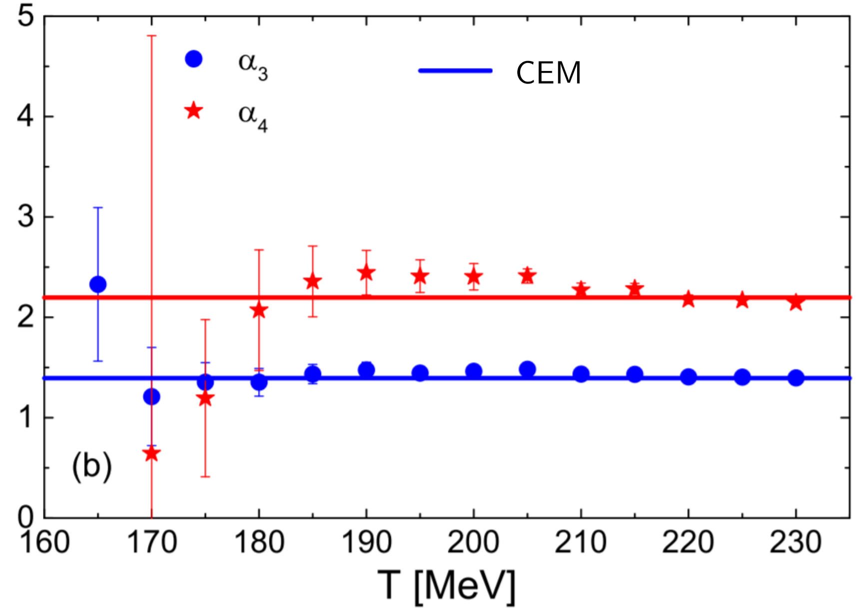
Regular behavior at real μ_B → *no-critical-point scenario*

CEM: Fourier coefficients

$b_k(T)$

$$\alpha_k(T) \equiv b_k(T)\,\frac{[b_1(T)]^{k-2}}{[b_2(T)]^{k-1}}$$



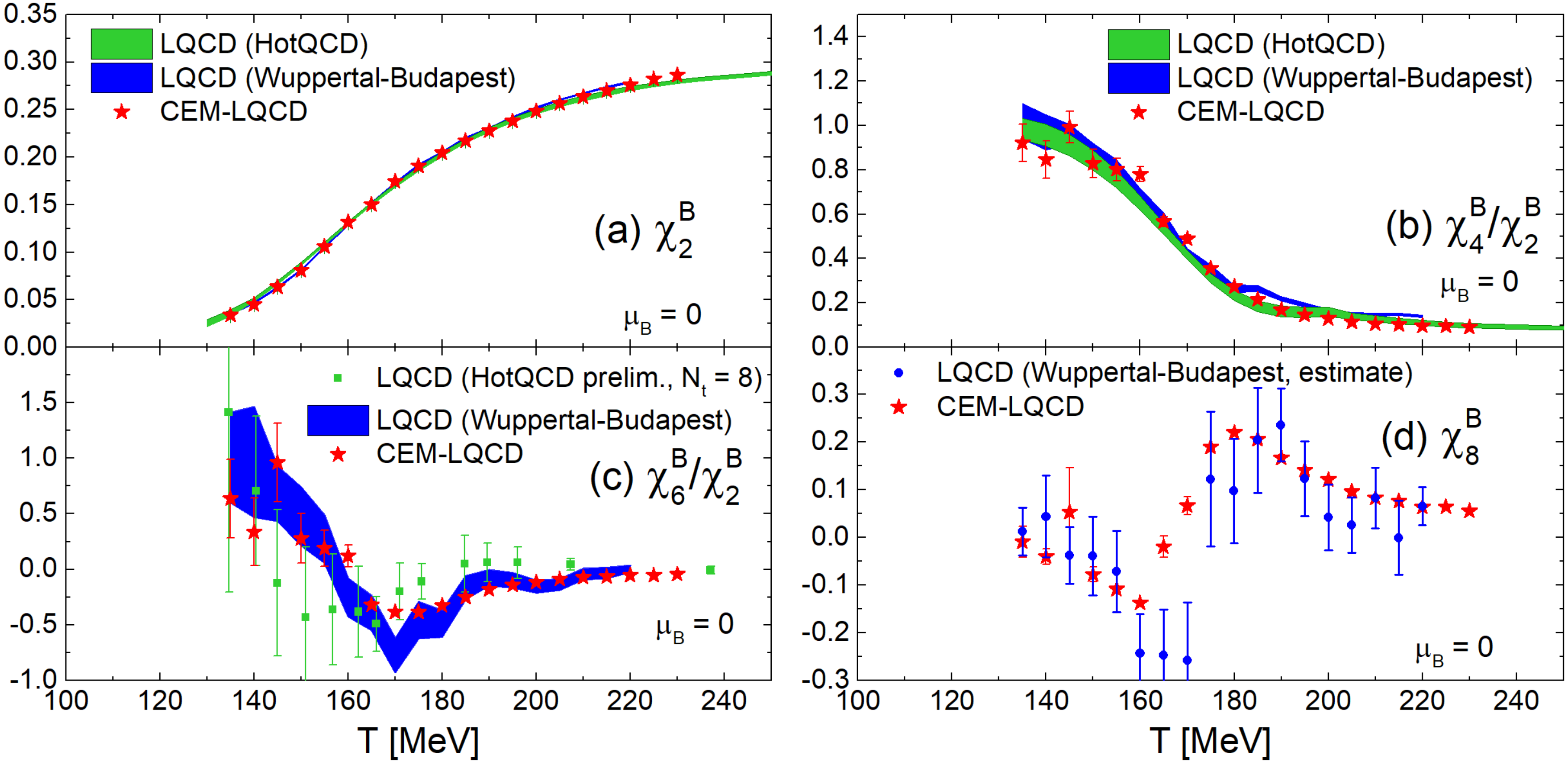


CEM: $b_1(T)$ and $b_2(T)$ as input → consistent description of $b_3(T)$ and $b_4(T)$

Lattice data on $b_{3,4}(T)$ inconclusive at $T \leq 170$ MeV

CEM: Baryon number susceptibilities

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2-k}(x_+) + (-1)^k\,\mathrm{Li}_{2-k}(x_-)\right] + 3\left[\mathrm{Li}_{4-k}(x_+) + (-1)^k\,\mathrm{Li}_{4-k}(x_-)\right]\right\}$$

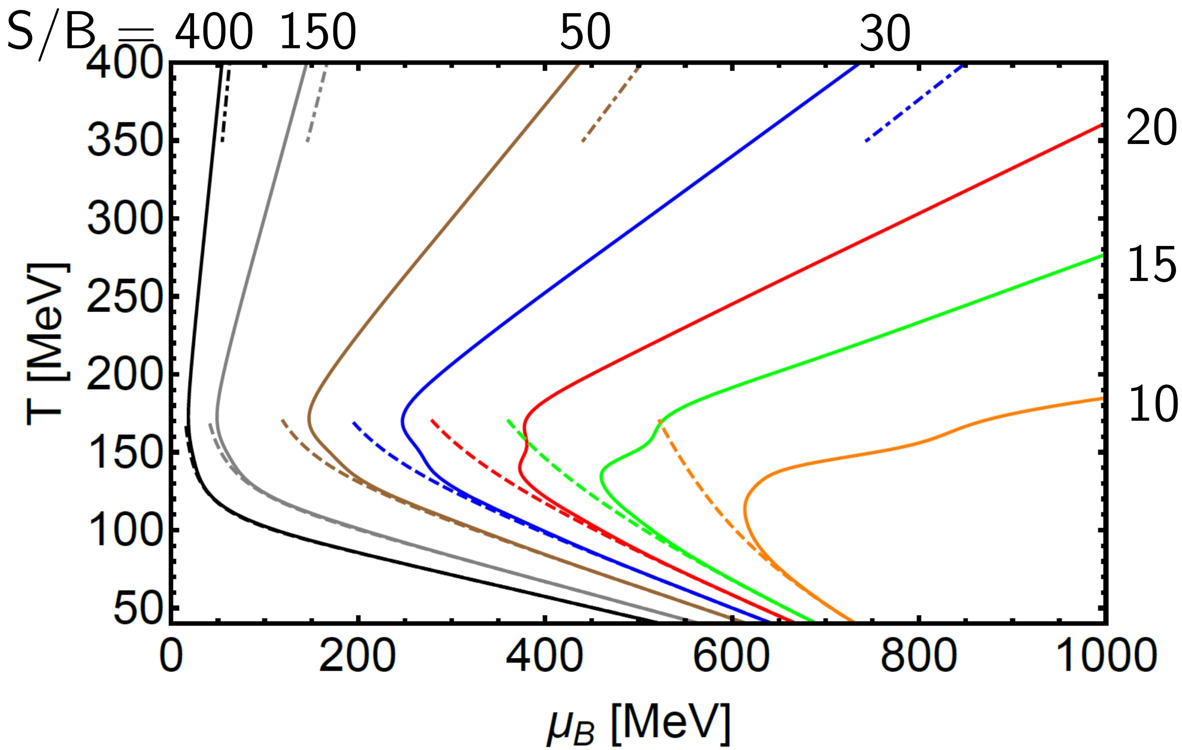


Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)

CEM: Equation of state

$$\frac{p(T,\mu_B)}{T^4} = \frac{p_0(T)}{2} - \frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_2(x_+) - \mathrm{Li}_2(x_-)\right] + 3\left[\mathrm{Li}_4(x_+) - \mathrm{Li}_4(x_-)\right]\right\}$$

Input: $p_0(T),\ b_{1,2}(T)$ $\leftarrow$ parametrized LQCD + HRG



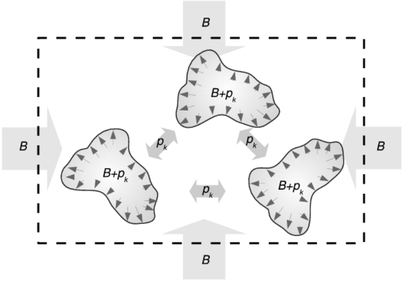
Tabulated CEM EoS available at https://fias.uni-frankfurt.de/~vovchenko/cem_table/

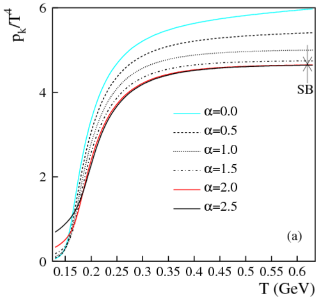
Currently restricted to single chemical potential (μ_B) and no critical point

Hagedorn (bag-like) resonance gas model with repulsive interactions

exactly solvable model with a (phase) transition between hadronic matter and QGP

[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; Ferroni, Koch, PRC '09]



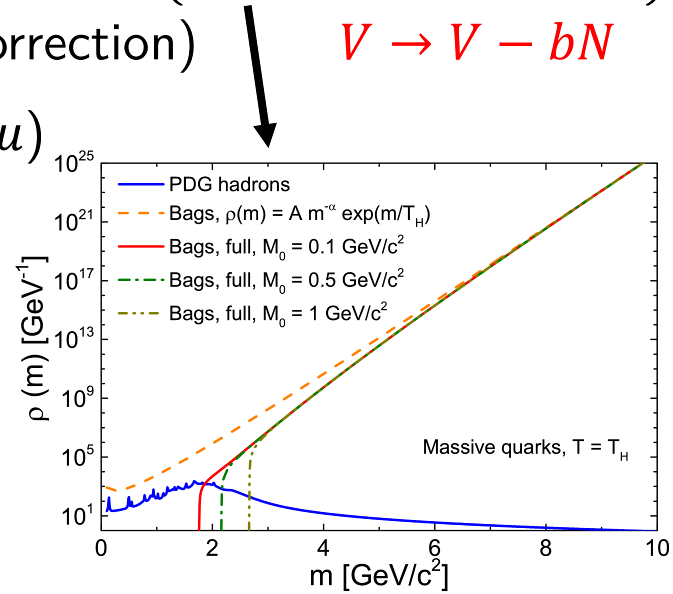


Here the model equation of state is constrained to lattice QCD

V.V., *M.I. Gorenstein, C. Greiner, H. Stoecker, Phys. Rev. C 99, 045204 (2019)*

Hagedorn bag-like model: formulation

- HRG + quark-gluon bags $\rho_Q(m,v) = C\, v^{\gamma}\,(m - Bv)^{\delta}\, \exp\left\{\frac{4}{3}[\sigma_Q]^{1/4}\, v^{1/4}\,(m-Bv)^{3/4}\right\}$
- Non-overlapping particles (excluded volume correction) $V \to V - bN$
- Isobaric (pressure) ensemble $(T, V, \mu) \to (T, s, \mu)$
- *Massive* (thermal) partons (**new element**)



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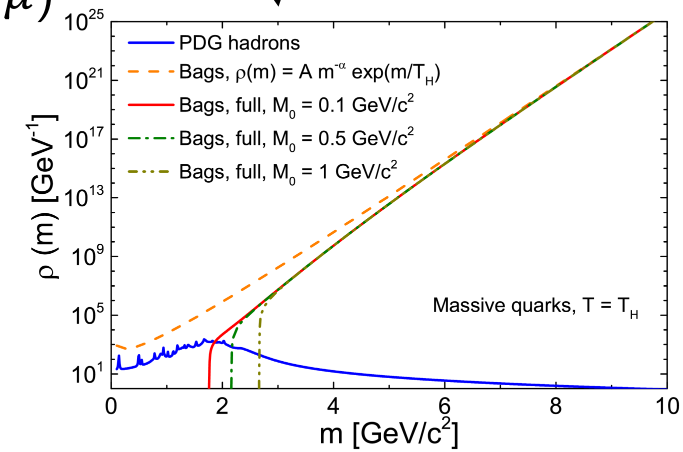
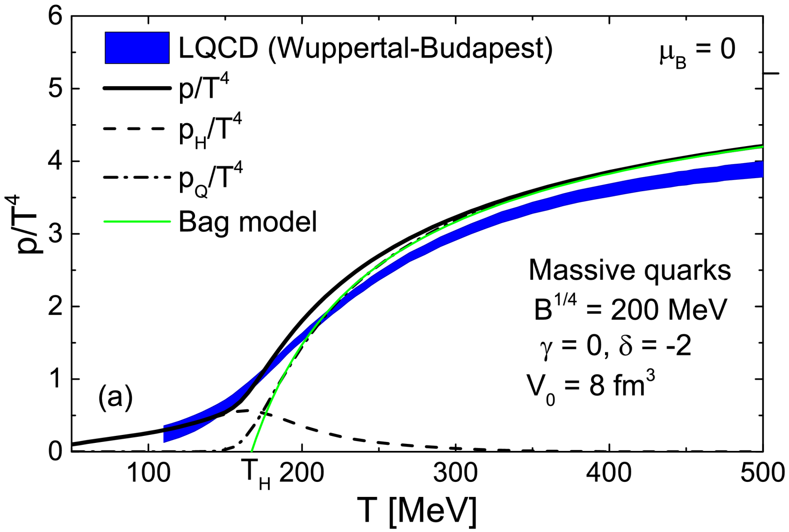
Resulting picture: transition (crossover, 1st order, 2nd order, etc.) between HRG and MIT bag model EoS, within single partition function



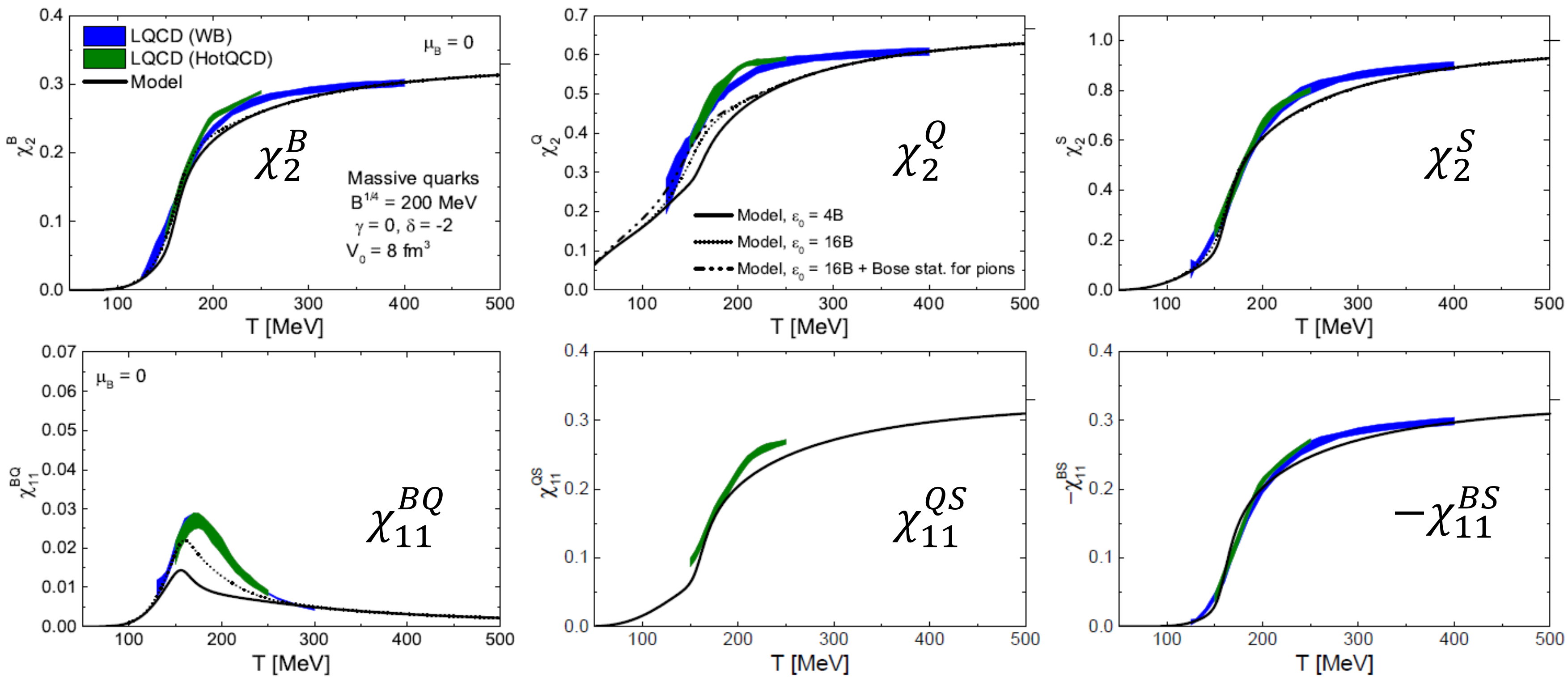


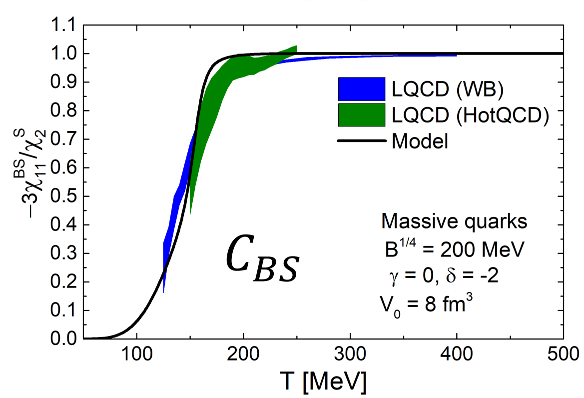
"Crossover" parameter set

$\gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3$
$m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV}$
$m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV}$

$T_H \simeq 167$ MeV

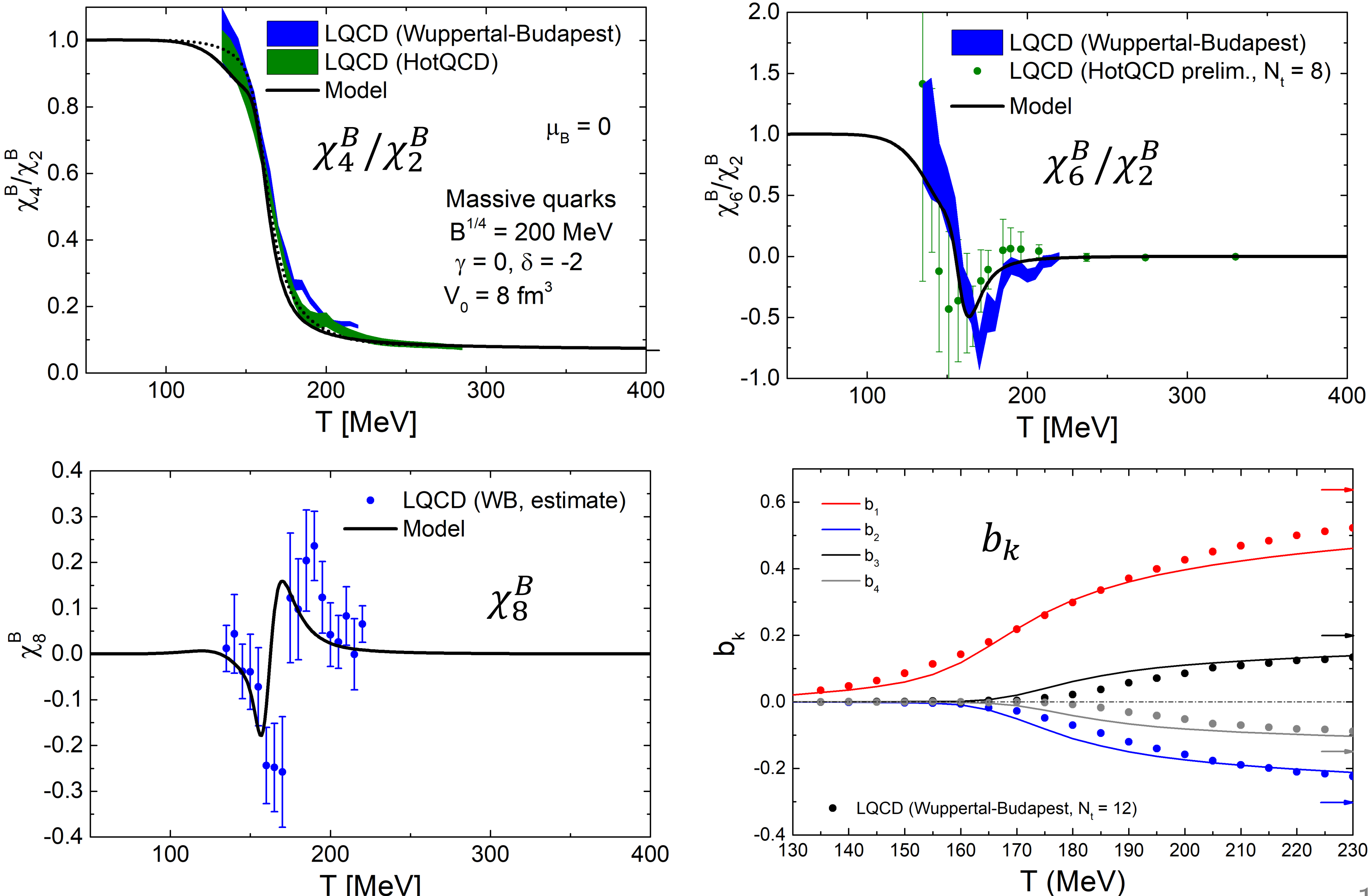
Hagedorn model: Susceptibilities



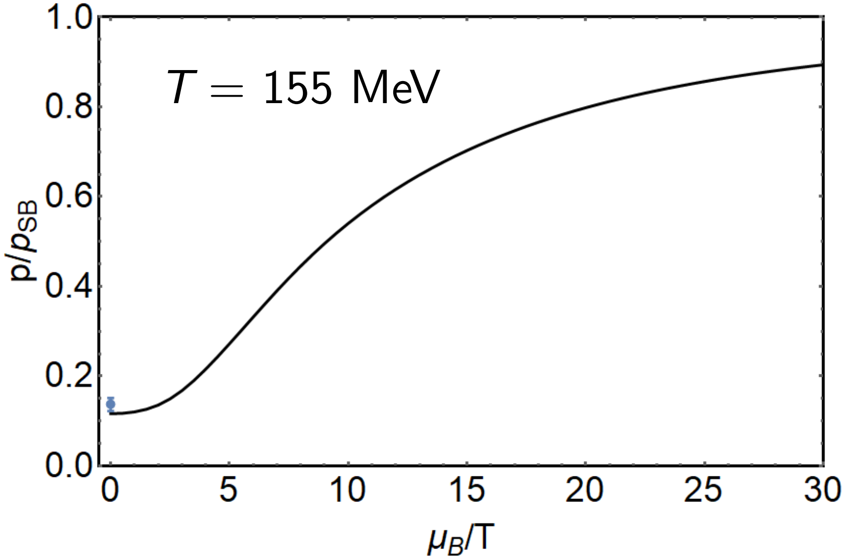


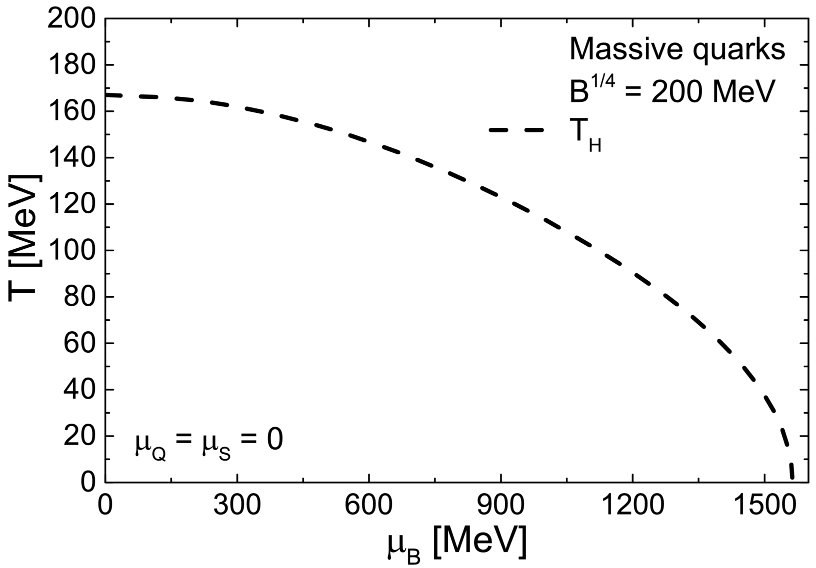
Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)

Hagedorn model: Susceptibilities and Fourier



Hagedorn model: Finite baryon density





- Crossover transition to a QGP-like phase in both the T and μ_B directions
- Essentially a built-in "switching" function between HRG and QGP, thermodynamically consistent by construction (single partition function)
- Critical point/phase transition at finite μ_B can be incorporated through μ_B-dependence of γ and δ exponents in bag spectrum

see Gorenstein, Gazdzicki, Greiner, Phys. Rev. C (2005)

More on this at SQM2019

Signatures of a critical point/phase transition at finite baryon density

currently no indications for the location of QCD critical point from lattice data, “small” $\mu_B/T \leq$ 2-3 disfavored

[Bazavov et al., 1701.04325; **V.V.** et al. 1711.01261; Fodor et al., 1807.09862]

Recent works incorporating a CP to study its signatures in heavy-ion collisions:

- P. Parotto et al., arxiv:1805.05249 – 3D Ising model, matched with LQCD susceptibilities, CP location can be varied
- C. Plumberg, T. Welle, J. Kapusta, arxiv:1812.01684 – CP through a switching function, location can be varied
- R. Critelli et al., arxiv:1706.00455 – holographic gauge/gravity corr., CP at “small” energies

This work: signatures of a critical point and a phase transition at finite density in the cluster expansion (imaginary μ_B LQCD observables)

A model with a phase transition

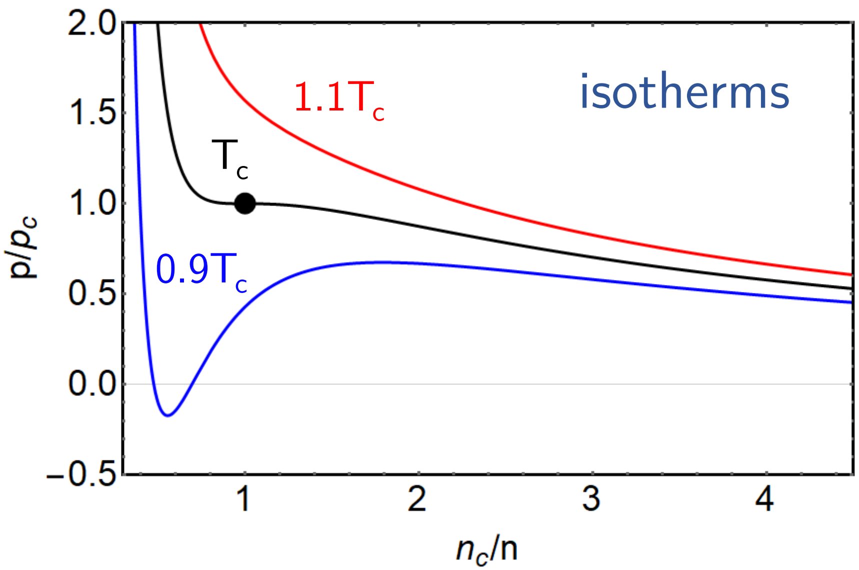
Our starting point is a single-component fluid. We are looking for a theory with a phase transition where Mayer's cluster expansion

$$\frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

can be worked out explicitly. The **"tri-virial" model (TVM)**

$$p(T,n) = T\,n + T\,\left(b - \frac{a}{T}\right)n^2 + T\,b^2\,n^3$$

which is the vdW equation truncated at n^3, has the required features.



Critical point:

$$\left(\frac{\partial p}{\partial n}\right)_T = 0, \qquad \left(\frac{\partial^2 p}{\partial n^2}\right)_T = 0$$

$$T_c = \frac{\sqrt{3}-1}{2}\,\frac{a}{b}, \qquad n_c = \frac{1}{\sqrt{3}\,b}, \qquad p_c = \frac{3-\sqrt{3}}{18}\,\frac{a}{b^2}$$

TVM in the grand canonical ensemble (GCE)

Transformation from (T, n) variables to (T, μ) [or (T, λ)] variables

$$p(T, n) = T\,n + T\left(b - \frac{a}{T}\right) n^2 + T\,b^2\,n^3$$

$$p(T, n) = -\left(\frac{\partial F}{\partial V}\right)_{T,N} \quad \Rightarrow \quad F(T, V, N) \quad \Rightarrow \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right], \qquad \lambda \equiv e^{\mu/T}$$

The defining transcendental equation for the GCE particle number density $n(T, \lambda)$

This equation encodes the analytic properties of the grand potential associated with a phase transition

TVM: the branch points

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right]$$

The defining equation permits multiple solutions therefore $n(T, \lambda)$ is multi-valued and has singularities – the **branch points:**

$$\left(\frac{\partial\lambda}{\partial n}\right)_T = 0 \qquad \Rightarrow \qquad 3(bn_{\mathrm{br}})^2 + 2\left(1 - \frac{a}{bT}\right) bn_{\mathrm{br}} + 1 = 0$$

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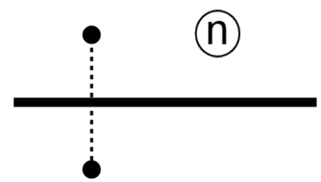
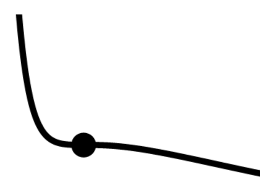
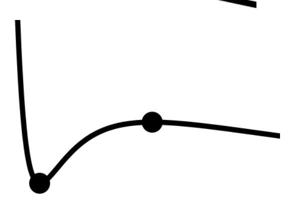
Solutions:

- $T > T_C$: two c.c. roots $n_{\rm br1} = (n_{\rm br2})^*$ 

crossover singularities

- $T = T_C$: $n_{\rm br1} = n_{\rm br2} = n_{\rm c}$ 
the critical point

- $T < T_C$: two real roots $n_{\rm sp1}$ and $n_{\rm sp2}$ 
the spinodal points

see also M. Stephanov, Phys. Rev. D 73, 094508 (2006)

TVM: Mayer's cluster expansion

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right] \quad \xrightarrow{?} \quad \frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

TVM: Mayer's cluster expansion

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right]$$

?

$$\frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

Lagrange inversion theorem

If $y=f(x)$, $y_0=f(x_0)$, $f'(x_0)\neq 0$, then

3.6.6

$$x=x_0+\sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}\left[\frac{d^{k-1}}{dx^{k-1}}\left\{\frac{x-x_0}{f(x)-y_0}\right\}^k\right]_{x=x_0}$$

from Abramowitz, Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables"

TVM: Mayer's cluster expansion

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right]$$

?

$$\frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

Lagrange inversion theorem

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$$y \equiv \lambda, \quad x \equiv n, \quad f(x) \equiv \lambda(n;T)$$

$$\lambda_0 = 0, \quad n_0 = 0$$

TVM: Mayer's cluster expansion

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right] \quad \overset{?}{\Longrightarrow} \quad \frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

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$$y \equiv \lambda, \quad x \equiv n, \quad f(x) \equiv \lambda(n;\,T)$$

$$\lambda_0 = 0, \quad n_0 = 0$$

Result:

$$b_k(T) = 2\,\frac{\phi(T)}{T^3}\,[b\,\phi(T)]^{k-1}\,\frac{1}{k!}\left(\frac{3k}{2}\right)^{\frac{k-1}{2}} \lim_{x\to 0}\frac{d^{k-1}}{dx^{k-1}}\exp\left[-2\sqrt{\frac{2k}{3}}\left(1-\frac{a}{bT}\right)x - x^2\right]$$

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Generating function of Hermite polynomials: $e^{2tx-\frac{1}{2}x^2} = \sum_{n=0}^{\infty} H_n(t)\,\frac{x^n}{n!}$

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$$b_k(T) = 2\,\frac{\phi(T)}{T^3}\,[b\,\phi(T)]^{k-1}\,\frac{1}{k!}\,\left(\frac{3k}{2}\right)^{\frac{k-1}{2}}\;H_{k-1}\left[-\sqrt{\frac{2k}{3}}\,\left(1-\frac{a}{b\,T}\right)\right]$$

A potentially non-trivial behavior of cluster integrals b_k associated with a presence of a phase transition is determined by the Hermite polynomials

Asymptotic behavior of cluster integrals

Asymptotic behavior of b_k determined mainly by Hermite polynomials

$$b_k \sim H_{k-1}\left[-\sqrt{\frac{2k}{3}}\left(1-\frac{a}{b\,T}\right)\right]$$

A caveat: both the argument and the index of H tend to large values.

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A caveat: both the argument and the index of H tend to large values. Such a case was analyzed in **[D. Dominici, arXiv:math/0601078]**

1) $x > \sqrt{2n}$ $\quad H_n(x) \overset{n\to\infty}{\simeq} \exp\left[\frac{x^2-\sigma x - n}{2} + n\ln(\sigma+x)\right]\sqrt{\frac{1}{2}\left(1+\frac{x}{\sigma}\right)}, \qquad \sigma = \sqrt{x^2-2n}$

$T<T_c$

2) $x \approx \sqrt{2n}$ $\quad H_n(x) \overset{n\to\infty}{\simeq} \exp\left[\frac{n}{2}\ln(2n) - \frac{3}{2}n + \sqrt{2n}\,x\right]\sqrt{2\pi}\,n^{1/6}\,\mathrm{Ai}\left[\sqrt{2}\,(x-\sqrt{2n})\,n^{1/6}\right]$

$T=T_c$

3) $|x| < \sqrt{2n}$ $\quad H_n\left[\sqrt{2n}\sin\theta\right] \overset{n\to\infty}{\simeq} \sqrt{\frac{2}{\cos\theta}}\exp\left\{\frac{n}{2}\left[\ln(2n)-\cos(2\theta)\right]\right\}\cos\left\{n\left[\frac{1}{2}\sin(2\theta)+\theta-\frac{\pi}{2}\right]+\frac{\theta}{2}\right\}$

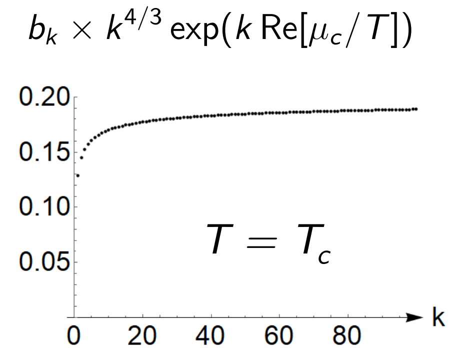
$T>T_c$

Asymptotic behavior changes as one traverses the critical temperature

Asymptotic behavior of cluster integrals

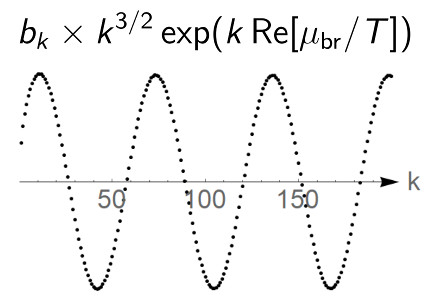
1) $T < T_c$: $$b_k(T) \overset{k\to\infty}{\simeq} A_- \frac{e^{-\frac{k\,\mu_{\rm sp1}}{T}}}{k^{3/2}}$$

b_k see the spinodal point of a first-order phase transition



2) $T = T_c$: $$b_k(T) \overset{k\to\infty}{\simeq} A_c \frac{e^{-\frac{k\,\mu_c}{T}}}{k^{4/3}}$$

b_k see the critical point



3) $T > T_c$: $$b_k(T) \overset{k\to\infty}{\simeq} A_+ \frac{e^{-\frac{k\,\mu^R_{\rm br}}{T}}}{k^{3/2}} \sin\left(k\,\frac{\mu^I_{\rm br}}{T} + \frac{\theta_0}{2}\right)$$

crossover singularities $\rightarrow$ oscillatory behavior of b_k

Behavior expected to be universal for the mean-field universality class, the likely effect of a change in universality class (e.g. 3D-Ising) is a modification of the power-law exponents

Applications to the QCD thermodynamics

TVM for "baryonic" pressure: $p_B(T,\mu) = T\,n_B + T\left(b - \frac{a}{T}\right) n_B^2 + T\,b^2\,n_B^3$

Symmetrization: $\mu_B \to -\mu_B$

$$p = p_B(T,\mu_B) + p_B(T,-\mu_B) + p_M(T)$$

"baryons" *"anti-baryons"* *"mesons"*

$$\frac{\rho_B(T, i\theta_B T)}{T^3} = i \sum_{k=1}^{\infty} b_k(T)\,\sin(k\,\theta_B T)$$

Cluster integrals become Fourier coefficients *(as long as $b_k(T) \overset{k\to\infty}{\to} 0$ holds)*

Riemann-Lebesgue lemma

Expected asymptotics

$$b_k(T) \overset{k\to\infty}{\simeq} A\,\frac{e^{-\frac{k\,\mu_{\rm br}^R}{T}}}{k^\alpha}\,\sin\left(k\,\frac{\mu_{\rm br}^I}{T} + \frac{\theta_0}{2}\right), \quad \frac{\mu_{\rm br}^R}{T} = \mathrm{Re}\left[\frac{\mu_{\rm br}}{T}\right], \quad \frac{\mu_{\rm br}^I}{T} = \mathrm{Im}\left[\frac{\mu_{\rm br}}{T}\right]$$

Can be tested in lattice QCD at imaginary chemical potential

Extracting information from Fourier coefficients

$$b_k(T) \overset{k\to\infty}{\simeq} A\,\frac{e^{-\frac{k\,\mu_{\rm br}^R}{T}}}{k^\alpha}\,\sin\left(k\,\frac{\mu_{\rm br}^I}{T}+\frac{\theta_0}{2}\right)$$

Real part of the limiting singularity determines the exponential suppression of Fourier coefficients

To extract $\mathrm{Re}[\mu_{\rm br}/T]$ fit b_k with $\log|b_k| = A - (3/2)\log k - k\,\mathrm{Re}\left[\frac{\mu_{\rm br}}{T}\right]$

Extracting information from Fourier coefficients

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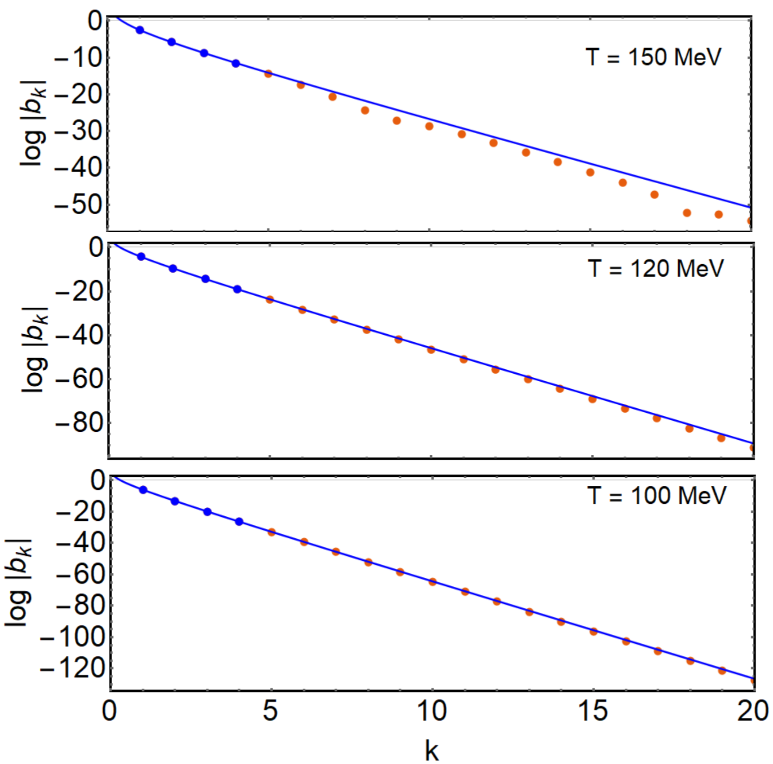
To extract Re[μ_{br}/T] fit b_k with $\log|b_k| = A - (3/2)\log k - k\,\mathrm{Re}\left[\frac{\mu_{br}}{T}\right]$

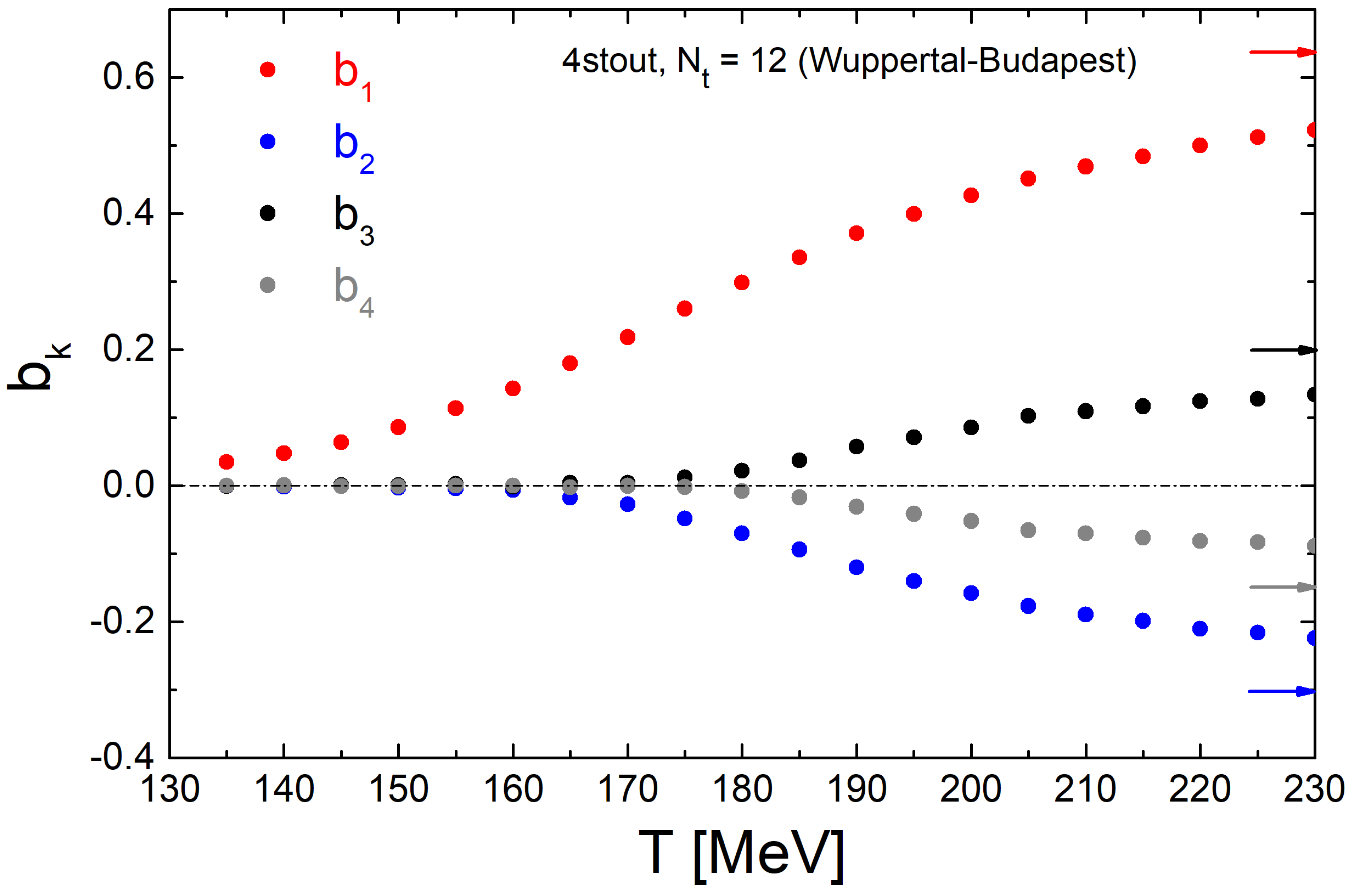
Illustration: TVM parameters fixed to a CP at $T_c = 120$ MeV, $\mu_c = 527$ MeV



Extracted Re[μ_{br}/T]

T [MeV]	Fit to b_1-b_4	True value
150	2.31	2.50
120	4.24	4.39
100	6.11	6.18

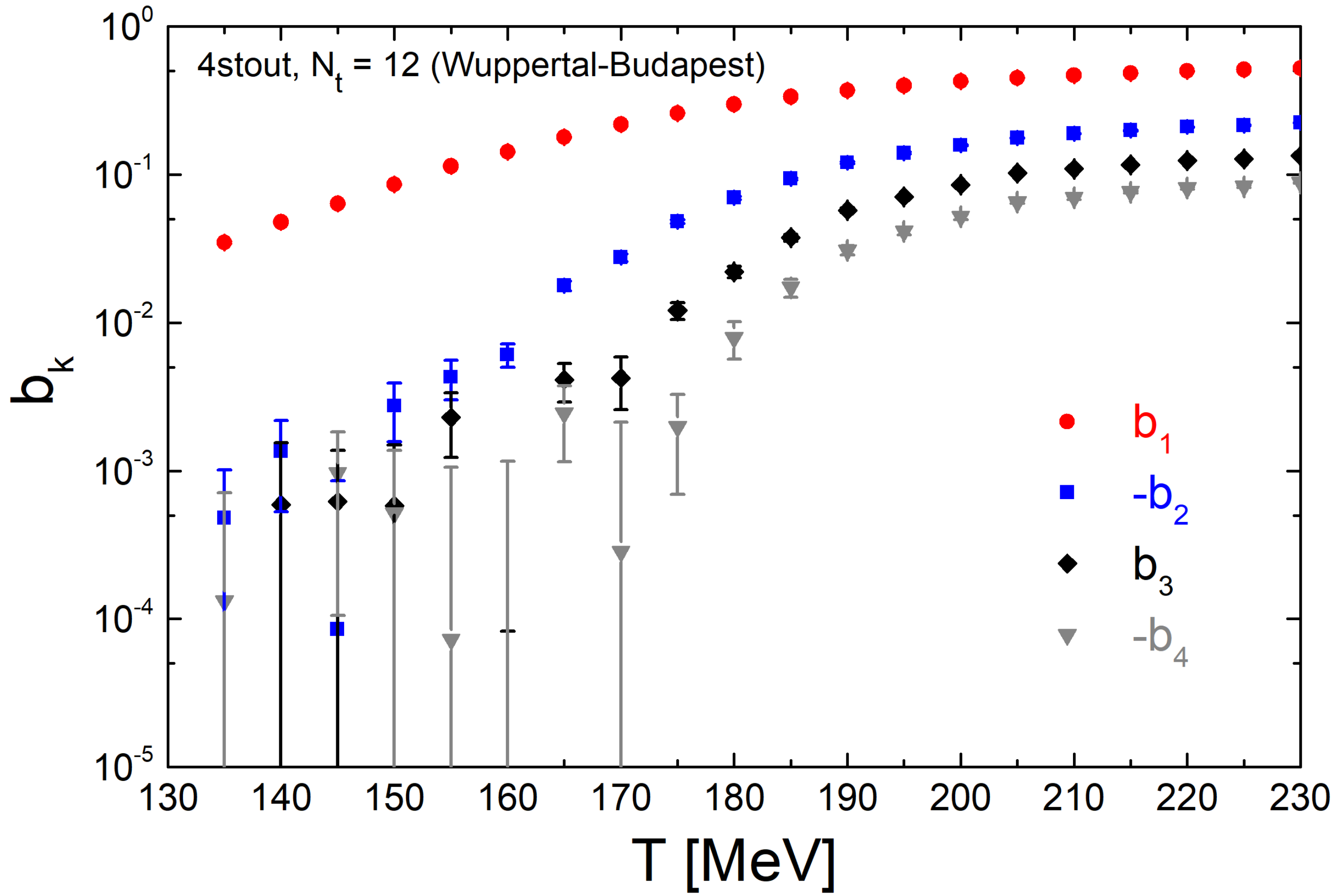
Fourier coefficients from lattice



Lattice QCD data (Wuppertal-Budapest), physical quark masses

[**V.V.**, Pasztor, Fodor, Katz, Stoecker, PLB 775, 71 (2017)]

Fourier coefficients from lattice



Lattice QCD data (Wuppertal-Budapest), physical quark masses

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Can one extract useful information from lattice data?

Extracting singularities from lattice data

Fit lattice data with an **ansatz:** $$\log|b_k| = A - \alpha \log k - k\,\mathrm{Re}\left[\frac{\mu_{\mathrm{br}}}{T}\right]$$

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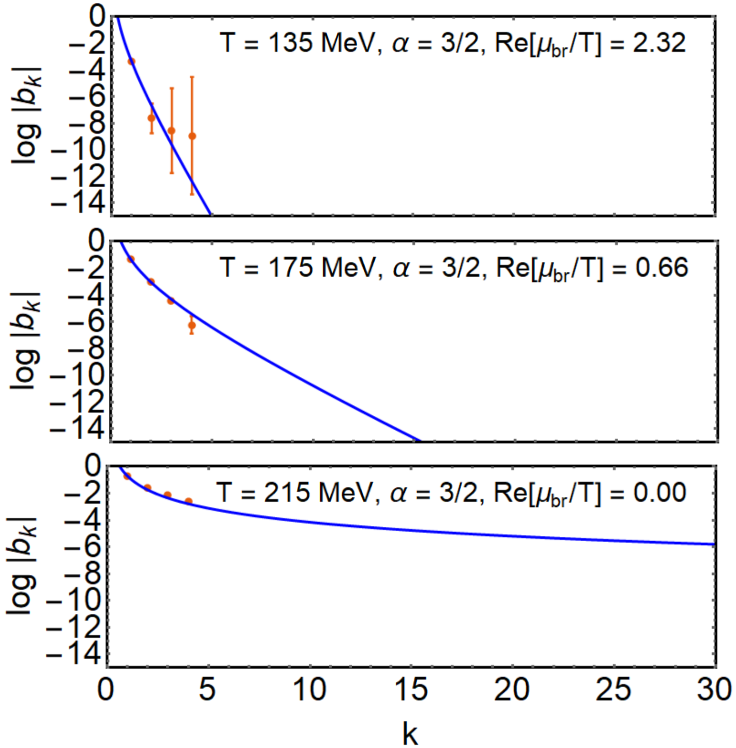

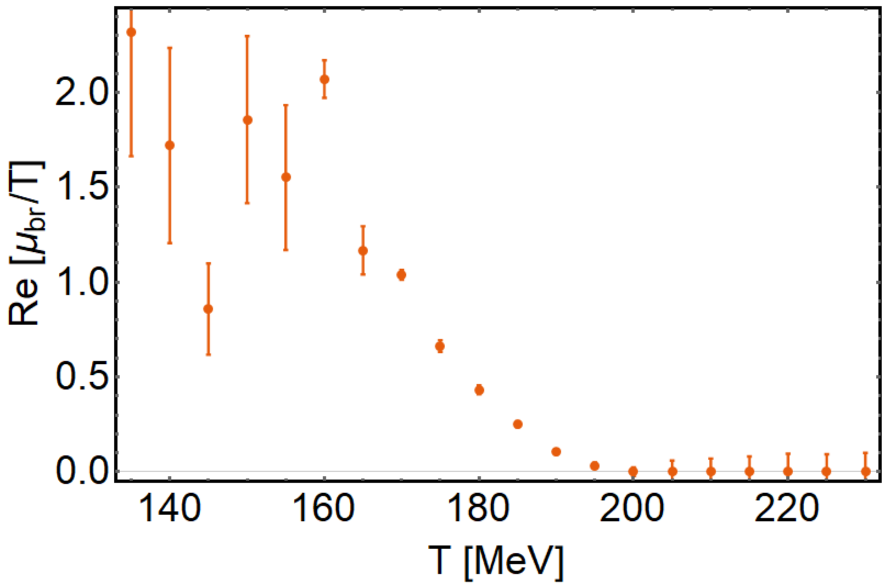

Quite similar results for $1 \leq \alpha \leq 2$

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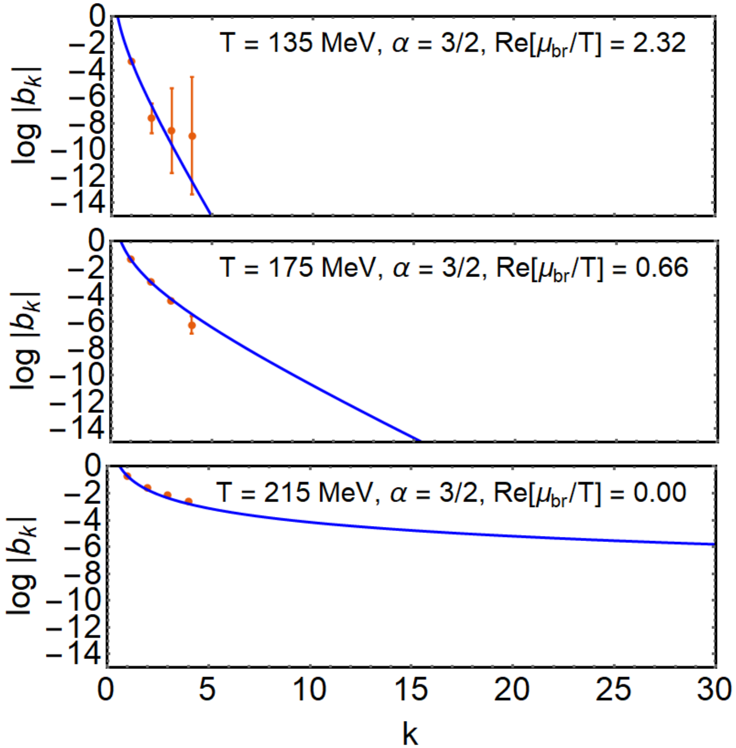

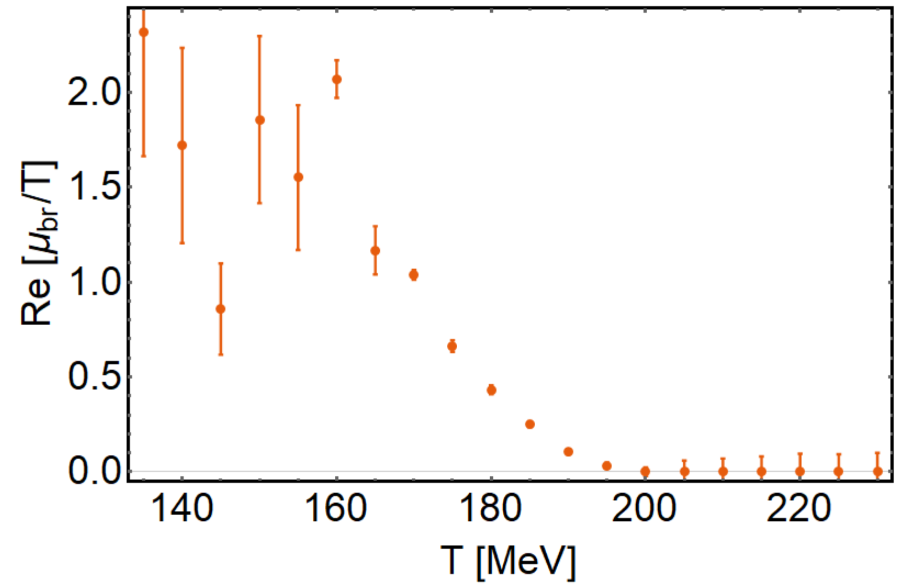

Quite similar results for $1 \leq \alpha \leq 2$

- $b_k \sim (-1)^{k-1}$ in the data $\rightarrow \quad \mathrm{Im}\left[\frac{\mu_{br}}{T}\right] \lesssim \pi$

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Fit lattice data with an **ansatz:**

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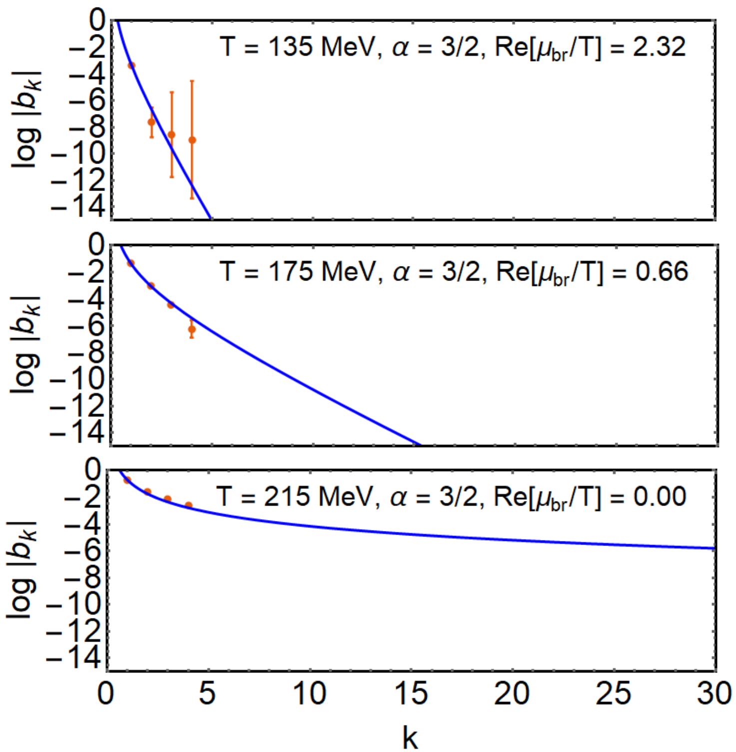

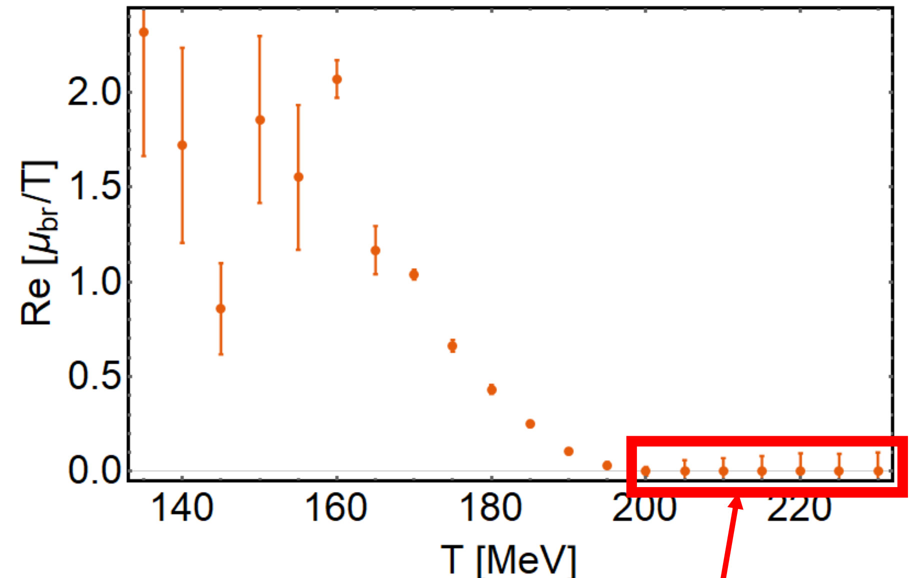

Quite similar results for $1 \leq \alpha \leq 2$

- $b_k \sim (-1)^{k-1}$ in the data $\rightarrow$ $\mathrm{Im}\left[\frac{\mu_{br}}{T}\right] \lesssim \pi$

- $\mathrm{Re}\left[\frac{\mu_{br}}{T}\right] \simeq 0$ for $T \gtrsim 200$ MeV $\rightarrow$ singularity at purely imaginary μ_B

Roberge-Weiss transition?

LQCD: $T_{RW} \sim 208$ MeV [C. Bonati et al., 1602.01426]

Summary

- Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate μ_B can be obtained in effective models constrained to all available lattice data, including *both* the Taylor expansion coefficients and Fourier coefficients of the cluster expansion

  Examples: Cluster Expansion Model, Hagedorn bag-like model, etc.

- Location of thermodynamic singularities, e.g. the QCD critical point, can be extracted from LQCD at imaginary chemical potential via exponential suppression of Fourier coefficients.

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Examples: Cluster Expansion Model, Hagedorn bag-like model, etc.

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Thanks for your attention!

Backup slides

Cluster expansion in fugacities

Expand in fugacity $\lambda_B = e^{\mu_B/T}$ instead of μ_B/T – a relativistic analogue of Mayer's cluster expansion:

$$\frac{p(T,\mu_B)}{T^4} = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_{|k|}(T)\, e^{k\mu_B/T} = \frac{p_0(T)}{2} + \sum_{k=1}^{\infty} p_k(T)\cosh(k\mu_B/T)$$

Net baryon density: $\frac{\rho_B(T,\mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T)\sinh(k\mu_B/T), \quad b_k \equiv k p_k$

Analytic continuation to imaginary μ_B yields trigonometric Fourier series

$$\frac{\rho_B(T,\mathrm{i}\tilde{\mu}_B)}{T^3} = \mathrm{i}\sum_{k=1}^{\infty} b_k(T)\,\sin\left(\frac{k\,\tilde{\mu}_B}{T}\right)$$

with Fourier coefficients $b_k(T) = \frac{2}{\pi\, T^4}\int_0^{\pi T} d\tilde{\mu}_B\, [\mathrm{Im}\,\rho_B(T,i\tilde{\mu}_B)]\,\sin(k\,\tilde{\mu}_B/T)$

Four leading coefficients b_k computed in LQCD at the physical point

[**V.V.**, A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852]

Why cluster expansion is interesting?

Convergence properties of cluster expansion determined by singularities of thermodynamic potential in complex fugacity plane → encoded in the asymptotic behavior of the Fourier coefficients b_k

Examples:

- ideal quantum gas $\quad b_k \sim (\pm 1)^{k-1} \frac{e^{-km/T}}{k^{3/2}}$ — *Bose-Einstein condensation*
- cluster expansion model $\quad b_k \sim (-1)^{k-1} \frac{|\lambda_{\rm br}|^{-k}}{k}$ — *$|\lambda_{br}| = 1$ → Roberge-Weiss transition at imaginary μ_B*

  [**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]
- excluded volume model $\quad b_k \sim (-1)^{k-1} \frac{|\lambda_{\rm br}|^{-k}}{k^{1/2}}$ — *No phase transition, but a singularity at a negative λ*

  [Taradiy, **V.V.**, Gorenstein, Stoecker, in preparation]
- chiral crossover $\quad b_k \sim \frac{e^{-k\tilde{\mu}_c}}{k^{2-\alpha}} \sin(k\theta_c + \theta_0)$ — *Remnants of chiral criticality at $\mu_B = 0$*

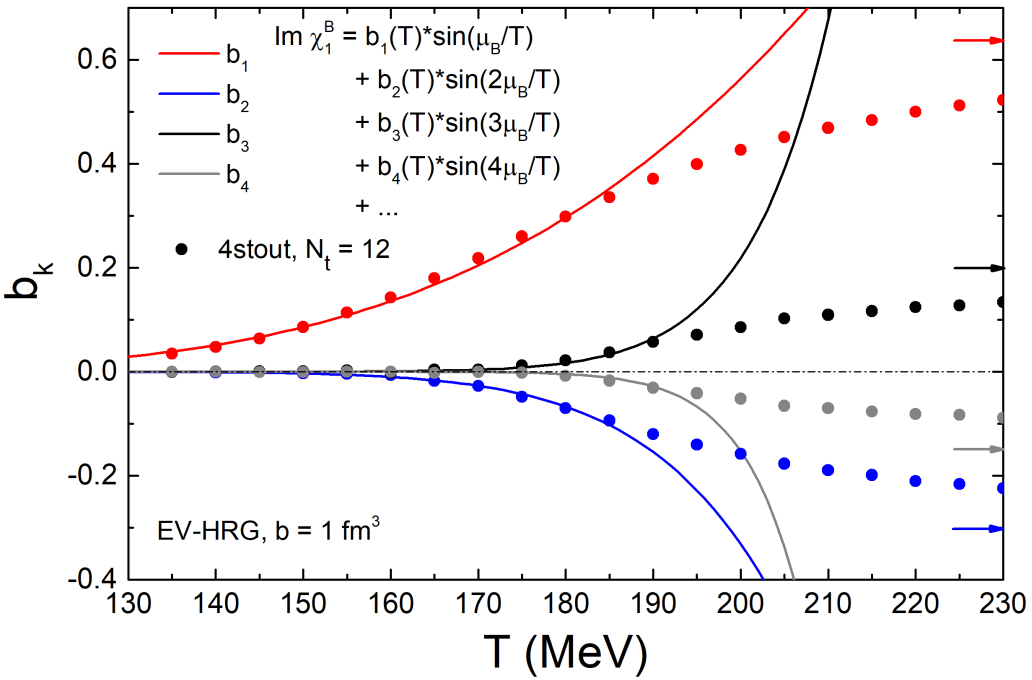
  [Almasi, Friman, Morita, Redlich, 1902.05457]

This work: signatures of a CP and a phase transition at finite density

HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) $V \to V - bN$

[Hagedorn, Rafelski, '80; Dixit, Karsch, Satz, '81; Cleymans et al., '86; Rischke et al., Z. Phys. C '91]



HRG with baryonic EV:

$$p_B(T, \mu_B) = p_B^{\mathrm{id}}(T, \mu_B - b p_B)$$

$$b_k^{ev}(T) = (-1)^{k-1} \frac{2\,k^k}{k!} (b\,T^3)^{k-1} \left[\frac{\phi_B(T)}{T^3}\right]^k$$

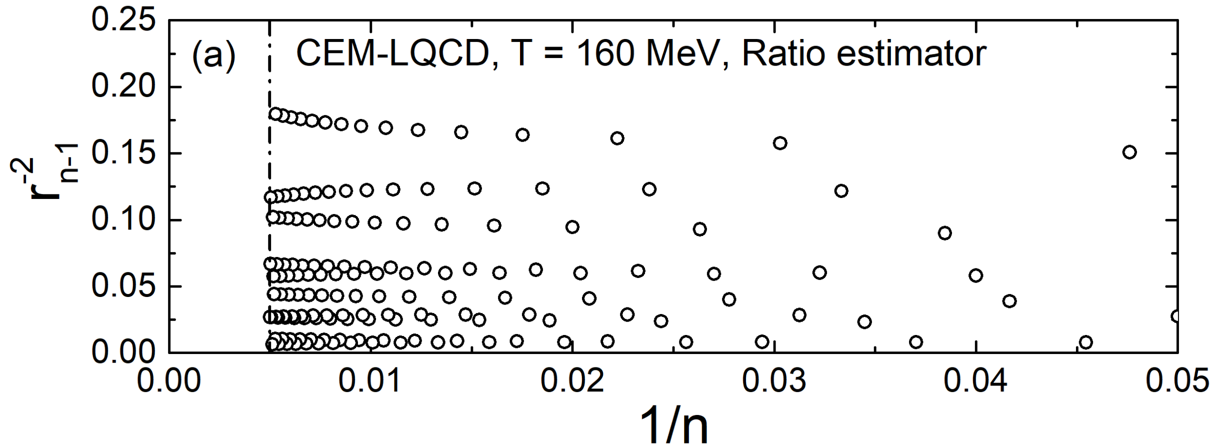
V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852

- Non-zero $b_k(T)$ for $k \geq 2$ signal deviation from ideal HRG
- EV interactions between baryons ($b \approx 1$ fm^3) reproduce lattice trend

Using estimators for radius of convergence

a) Ratio estimator:

$$r_n = \left| \frac{(2n+2)(2n+1)\chi^B_{2n}}{\chi^B_{2n+2}} \right|^{1/2}$$

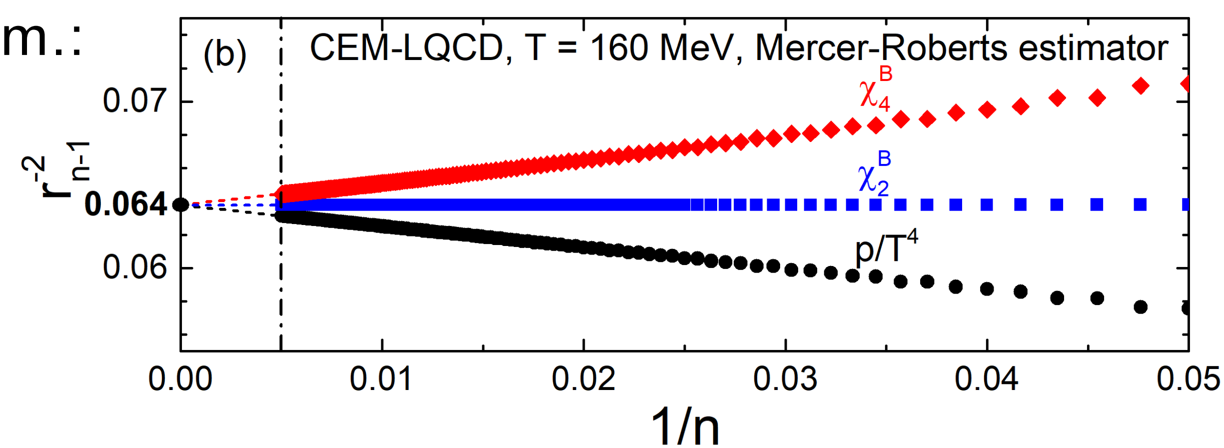


Ratio estimator is *unable* to determine the radius of convergence, nor to provide an upper or lower bound

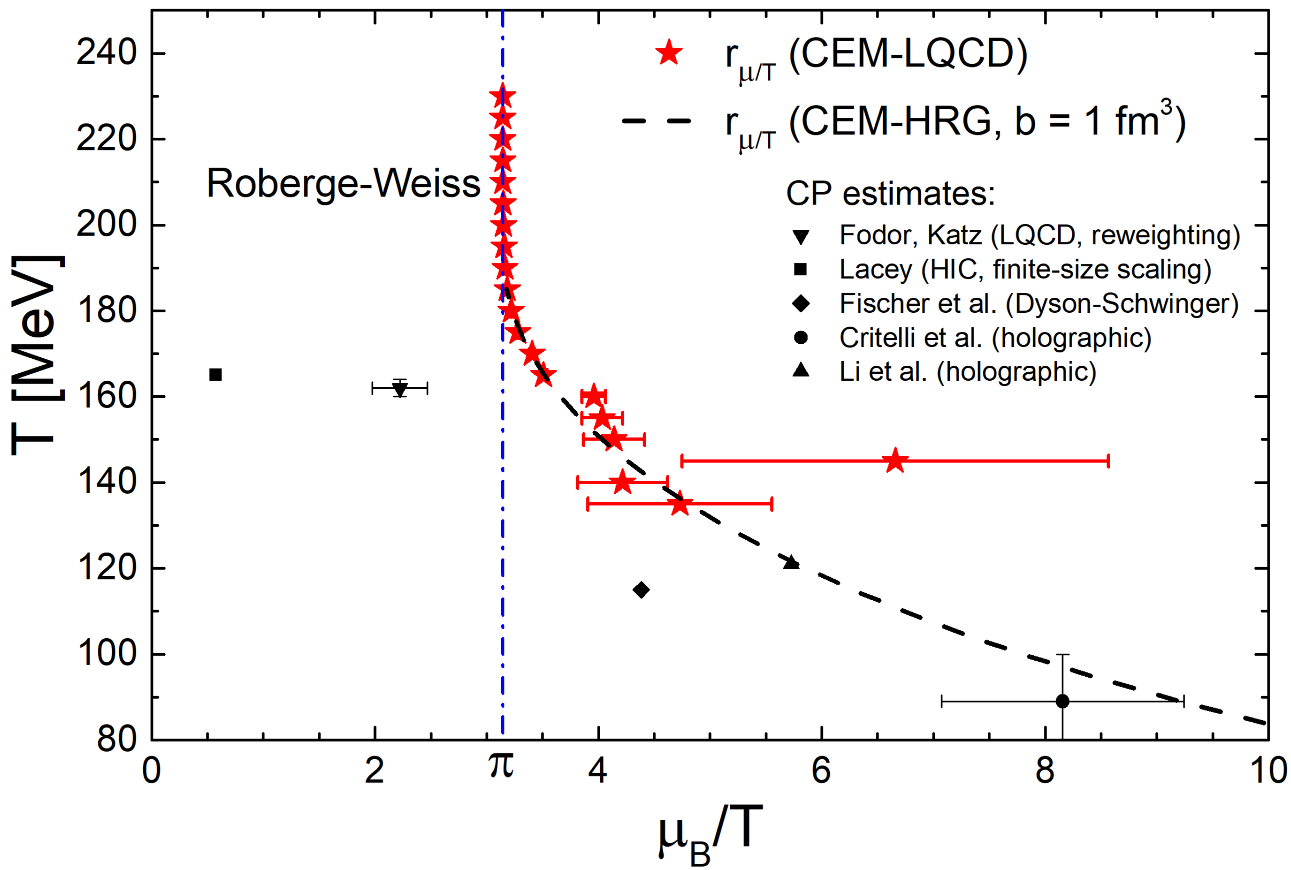
b) Mercer-Roberts estim.:

$$r_n = \left| \frac{c_{n+1}\, c_{n-1} - c_n^2}{c_{n+2}\, c_n - c_{n+1}^2} \right|^{1/4}$$

$$c_n = \frac{\chi^B_{2n}}{(2n)!}$$



CEM: Radius of convergence



Radius of convergence approaches Roberge-Weiss transition value

- At $T > T_{RW}$ expected $\left[\frac{\mu_B}{T}\right]_c = \pm i\pi$ [Roberge, Weiss, NPB '86] $T_{RW} \sim 208$ MeV [C. Bonati et al., 1602.01426]
- Complex plane singularities interfere with the search for CP

Expected asymptotics

- At low T/densities QCD $\simeq$ ideal hadron resonance gas

$$\frac{p^{\rm hrg}(T,\mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2\,\frac{\phi_B(T)}{T^3}\cosh\left(\frac{\mu_B}{T}\right),$$

$$\phi_B(T) = \sum_{i\in B}\int dm\,\rho_i(m)\,\frac{d_i\,m^2\,T}{2\pi^2}\,K_2\left(\frac{m}{T}\right),$$

$$p_0^{hrg}(T) = \frac{\phi_M(T)}{T^3}, \quad p_1^{hrg}(T) = \frac{2\,\phi_B(T)}{T^3}, \quad p_k^{\rm hrg}(T) \equiv 0,\ k \geq 2$$

- At high T QCD $\simeq$ ideal gas of massless quarks and gluons

$$\frac{p^{\rm SB}(T,\mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s}\left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_B}{3T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_B}{3T}\right)^4\right],$$

$$p_0^{\rm SB} = \frac{64\pi^2}{135}, \quad p_k^{\rm SB} = \frac{(-1)^{k+1}}{k^2}\,\frac{4\,[3+4\,(\pi k)^2]}{27\,(\pi k)^2}, \quad b_k^{SB} = k\,p_k^{\rm SB}.$$

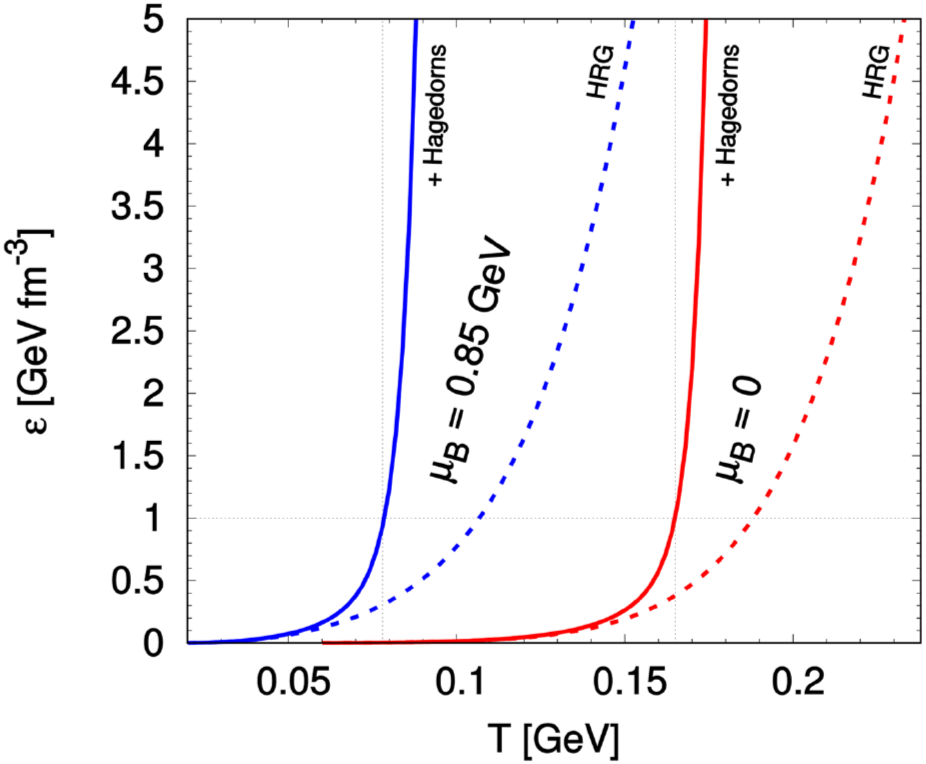
Lattice data explore intermediate, transition region $130 < T < 230$ *MeV*

*In this study we assume that $\mu_S = \mu_Q = 0$

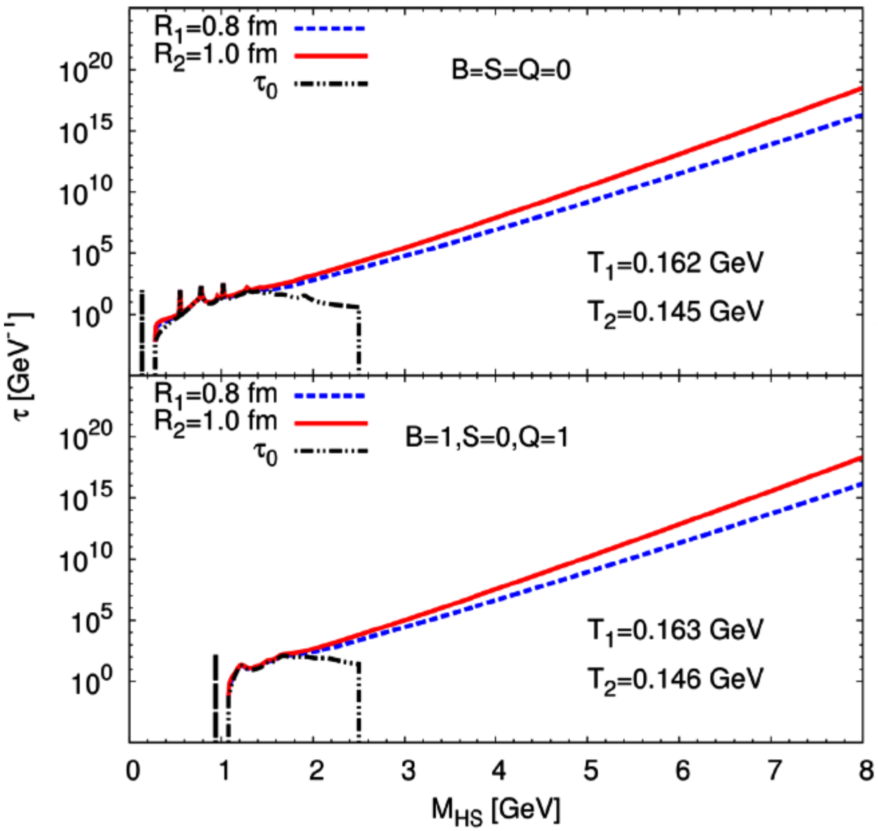
Hagedorn resonance gas

HRG + exponential Hagedorn mass spectrum, e.g. as obtained from the bootstrap equation [Hagedorn '65; Frautschi, '71]

$$\rho(m) = A\, m^{-\alpha} \exp(m/T_H)$$



[Beitel, Gallmeister, Greiner, 1402.1458]

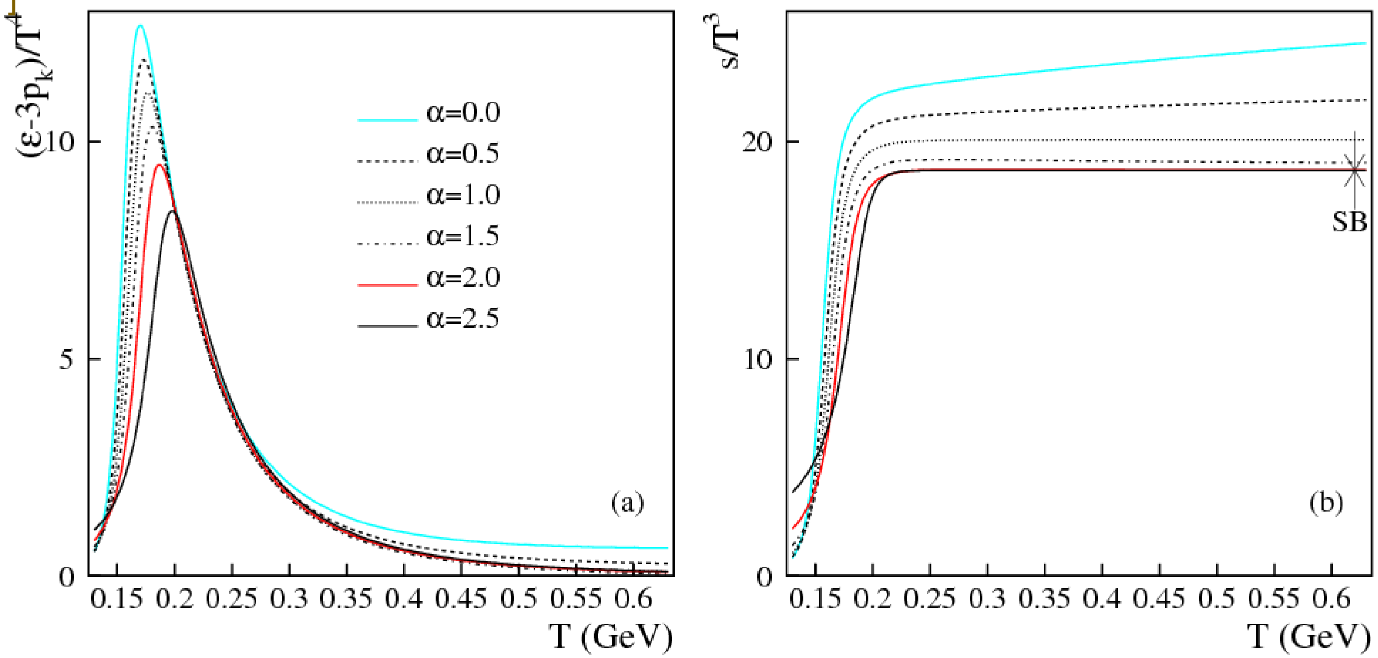


If Hagedorns are point-like, *T_H is the limiting temperature*

From limiting temperature to crossover

- A gas of extended objects → excluded volume
- Exponential spectrum of *compressible* QGP bags
- Both phases described by single partition function

[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; I. Zakout et al., NPA '07]



[Ferroni, Koch, PRC 79, 034905 (2009)]

Crossover transition in bag-like model qualitatively compatible with LQCD

Model formulation

Thermodynamic system of known hadrons and quark-gluon bags

Mass-volume density $\rho(m, v; \lambda_B, \lambda_Q, \lambda_S) = \rho_H + \rho_Q$

$$\rho_H(m, v; \lambda_B, \lambda_Q, \lambda_S) = \sum_{i \in \mathrm{HRG}} \lambda_B^{b_i} \lambda_Q^{q_i} \lambda_S^{s_i} d_i \, \delta(m - m_i) \, \delta(v - v_i)$$ **PDG hadrons**

$$\rho_Q(m, v; \lambda_B, \lambda_Q, \lambda_S) = C \, v^\gamma \, (m - Bv)^\delta \, \exp\left\{ \frac{4}{3} [\sigma_Q v]^{1/4} \, (m - Bv)^{3/4} \right\} \theta(v - V_0) \, \theta(m - Bv)$$

Quark-gluon bags [J. Kapusta, PRC '81; Gorenstein+, ZPC '84]

Non-overlapping particles → **isobaric (pressure) ensemble** [Gorenstein, Petrov, Zinovjev, PLB '81]

$$\hat{Z}(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int_0^\infty Z(T, V, \lambda_B, \lambda_Q, \lambda_S) \, e^{-sV} \, dV = [s - f(T, s, \lambda_B, \lambda_Q, \lambda_S)]^{-1}$$

$$f(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int dv \int dm \, \rho(m, v; \lambda_B, \lambda_Q, \lambda_S) \, e^{-v\, s} \, \phi(T, m)$$

The system pressure is $p = Ts^*$ with s^* being the *rightmost* singularity of $\hat{Z}$

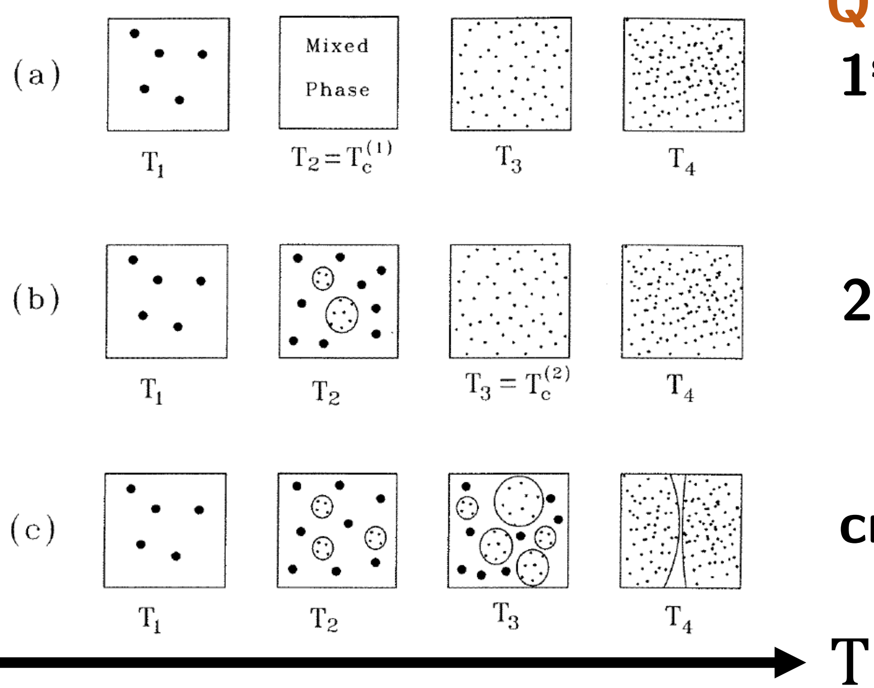
Mechanism for transition to QGP

The isobaric partition function, $\hat{Z}(T,s,\lambda) = [s - f(T,s,\lambda)]^{-1}$, has

- pole singularity $s_H = f(T, s_H, \lambda)$ **"hadronic" phase**
- singularity s_B in the function $f(T,s,\lambda)$ due to the exponential spec

$$p_B = T\, s_B = \frac{\sigma_Q}{3} T^4 - B$$

MIT bag model EoS for QGP [Chodos+, PRD '74; Baacke, APPB '77]



1st order PT "collision" of singularities $s_H(T_C) = s_B(T_C)$

2nd order PT

crossover $s_H(T) > s_B(T)$ at all T

Crossover transition

Type of transition is determined by exponents γ and δ of bag spectrum

Crossover seen in lattice, realized in model for $\gamma + \delta \geq -3$ and $\delta \geq -7/4$

[Begun, Gorenstein, W. Greiner, JPG '09]

Transcendental equation for pre

$$p(T, \lambda_B, \lambda_Q, \lambda_S) = T \sum_{i \in \mathrm{HRG}} d_i \, \phi(T, m) \, \lambda_B^{b_i} \, \lambda_Q^{q_i} \, \lambda_S^{s_i} \exp\left(-\frac{m_i p}{4BT}\right)$$

$$+ \frac{C}{\pi} T^{5+4\delta} [\sigma_Q]^{\delta+1/2} [B + \sigma_Q T^4]^{3/2} \left(\frac{T}{p - p_B}\right)^{\gamma+\delta+3} \Gamma\left[\gamma + \delta + 3, \frac{(p - p_B) V_0}{T}\right]$$

Solved numerically

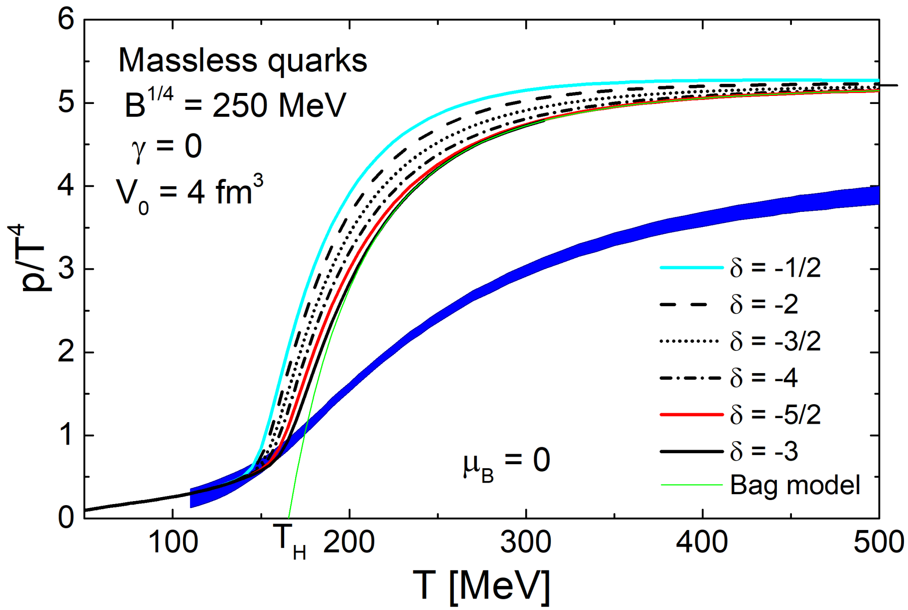
Calculation setup:

$$\gamma = 0, \quad -3 \leq \delta \leq -\frac{1}{2}, \quad B^{1/4} = 250 \text{ MeV}, \quad C = 0.03 \text{ GeV}^{-\delta+2}, \quad V_0 = 4 \text{ fm}^3$$

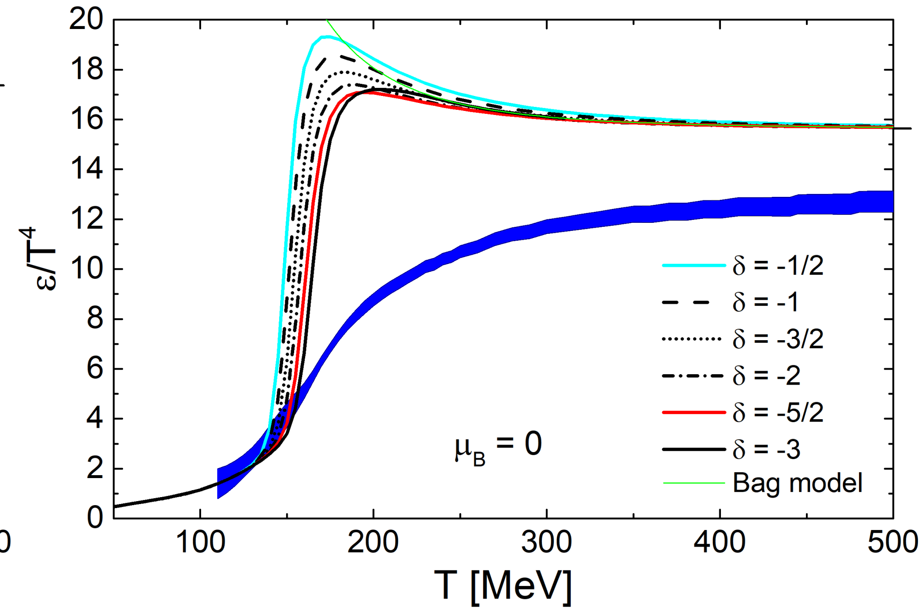
$$T_H = \left(\frac{3B}{\sigma_Q}\right)^{1/4} \simeq 165 \text{ MeV}$$

Thermodynamic functions

Pressure p/T^4



Energy density ε/T^4

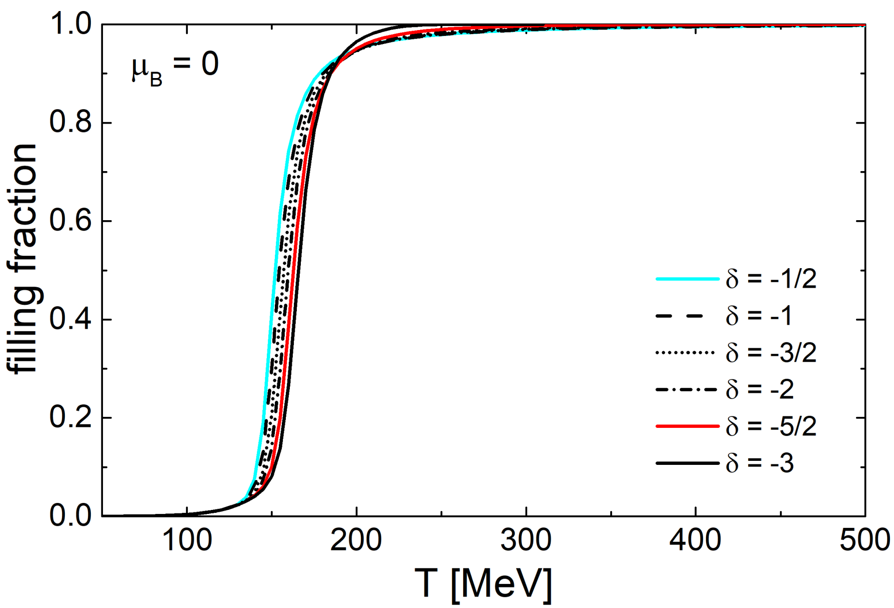


Lattice data from 1309.5258 (Wuppertal-Budapest)

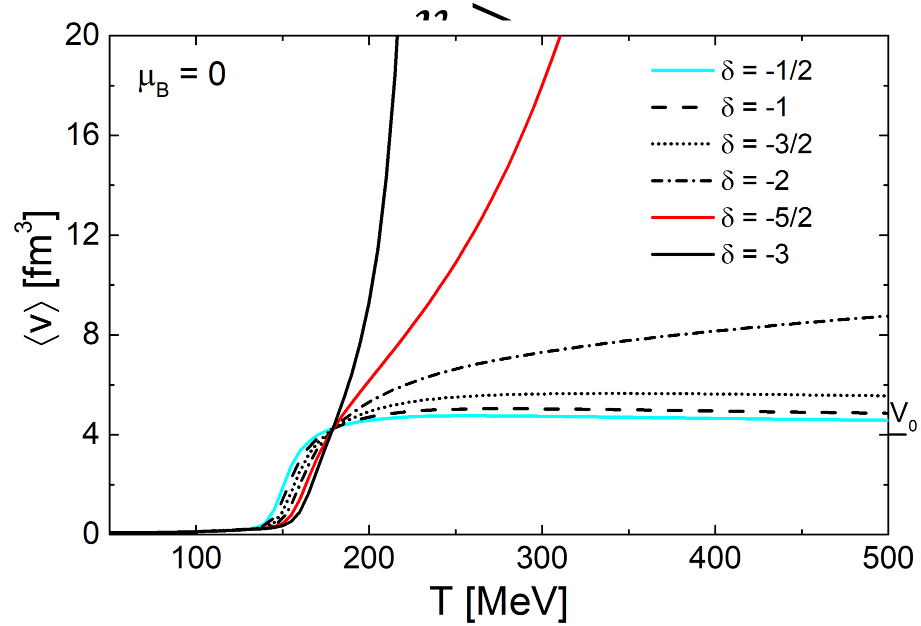
- Crossover transition towards bag model EoS
- Dependence on δ is mild
- Approach to the Stefan-Boltzmann limit is too fast
- Peak in energy density, not seen on the lattice

Nature of the transition

Filling fraction $= \frac{<V_{had}>}{V}$



Mean hadron volume <

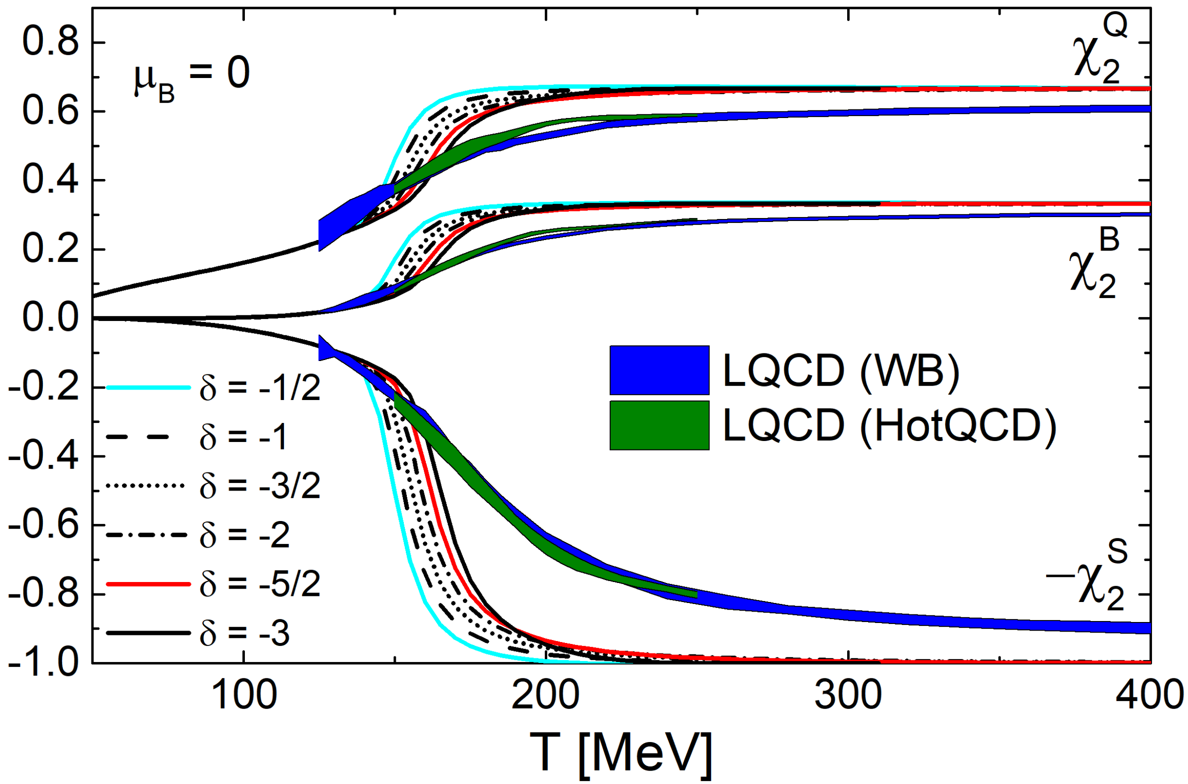


- Bags occupy almost whole space at large temperatures
- Strongest changes take place in the vicinity of T_H
- At $\delta < -7/4$ and $T \to \infty$ whole space — large bags with QGP

Conserved charges susceptibilities

$$\chi^{BSQ}_{lmn} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l\, \partial(\mu_S/T)^m\, \partial(\mu_Q/T)^n}$$

Available from lattice QCD, not considered in this type of model before



Qualitatively compatible with lattice QCD

Bag model with massive quarks

Main source of quantitative disagreement comes from inaccuracy of the standard MIT bag model with massless quarks for describing QGP

Quasiparticle models suggest sizable thermal masses of quarks and gluons in high-temperature QGP[Peshier et al., PLB '94; PRC '00; PRC '02]

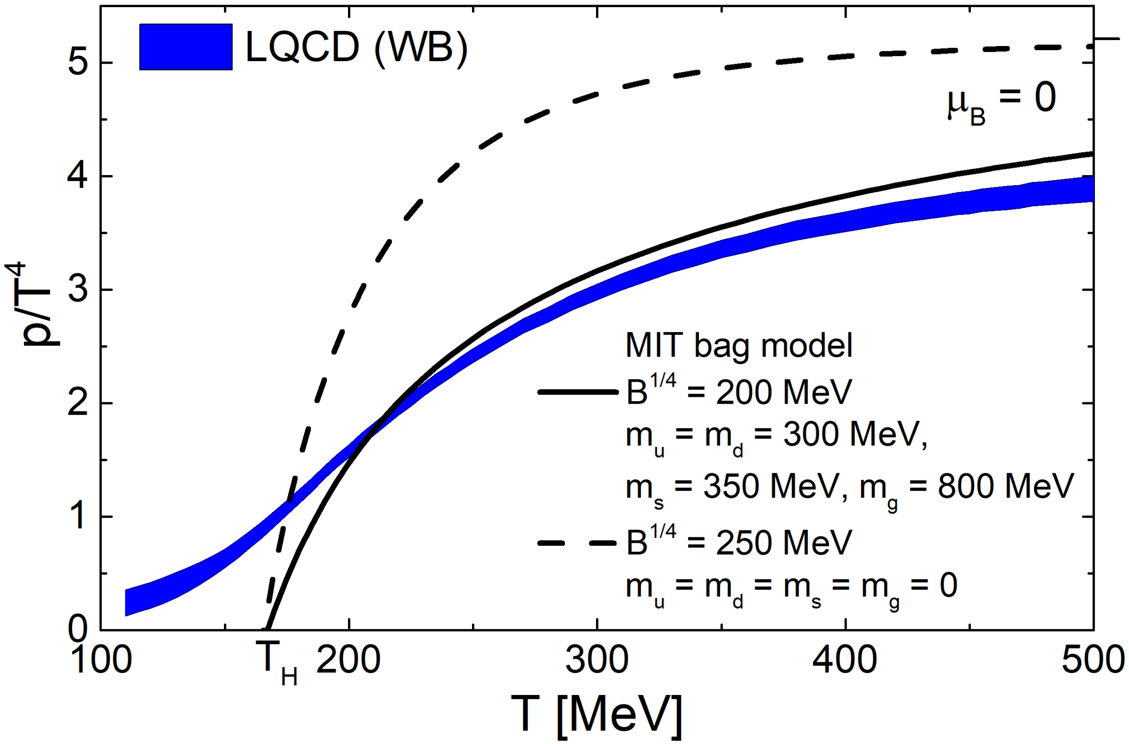
Heavy-bag model: bag model EoS with non-interacting *massive* quarks and gluons and the bag constant[Ivanov et al., PRC 72, 025804 (2005)]

Massive quarks and gluons instead of massless ones:

$$\sigma_Q(T,\lambda_B,\lambda_Q,\lambda_S) = \frac{8}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_g^2}}\left[\exp\left(\frac{\sqrt{k^2+m_g^2}}{T}\right)-1\right]^{-1}$$
$$+\sum_{f=u,d,s}\frac{3}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_f^2}}\left[\lambda_f^{-1}\exp\left(\frac{\sqrt{k^2+m_f^2}}{T}\right)+1\right]^{-1}$$
$$+\sum_{f=u,d,s}\frac{3}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_f^2}}\left[\lambda_f\exp\left(\frac{\sqrt{k^2+m_f^2}}{T}\right)+1\right]^{-1}$$

Bag model with massive quarks

Introduction of constituent masses leads to much better description of QGP



Parameters for the crossover model:

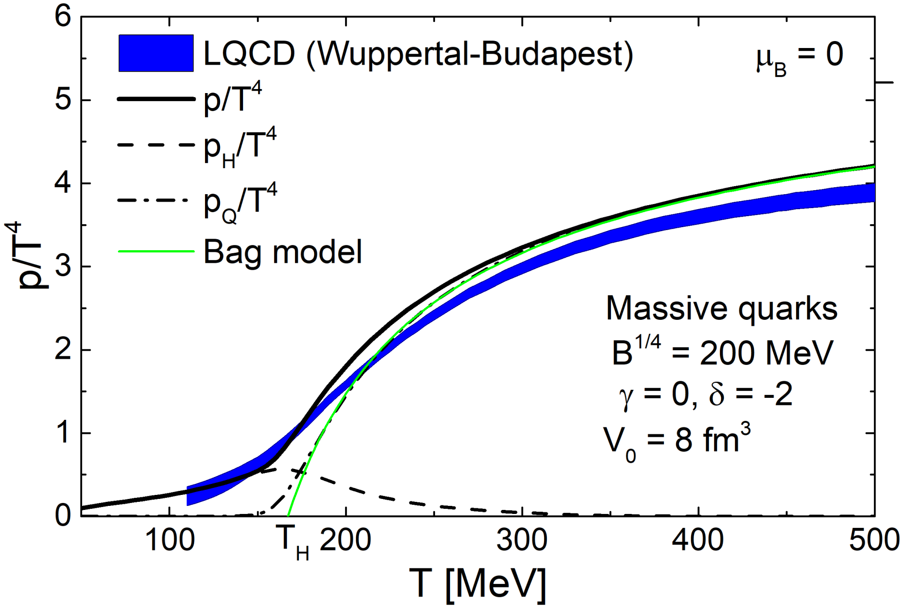
$$m_u = m_d = 300\ \text{MeV}, \quad m_s = 350\ \text{MeV}, \quad m_g = 800\ \text{MeV}, \quad B^{1/4} = 200\ \text{MeV}$$

$$\gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8\ \text{fm}^3$$

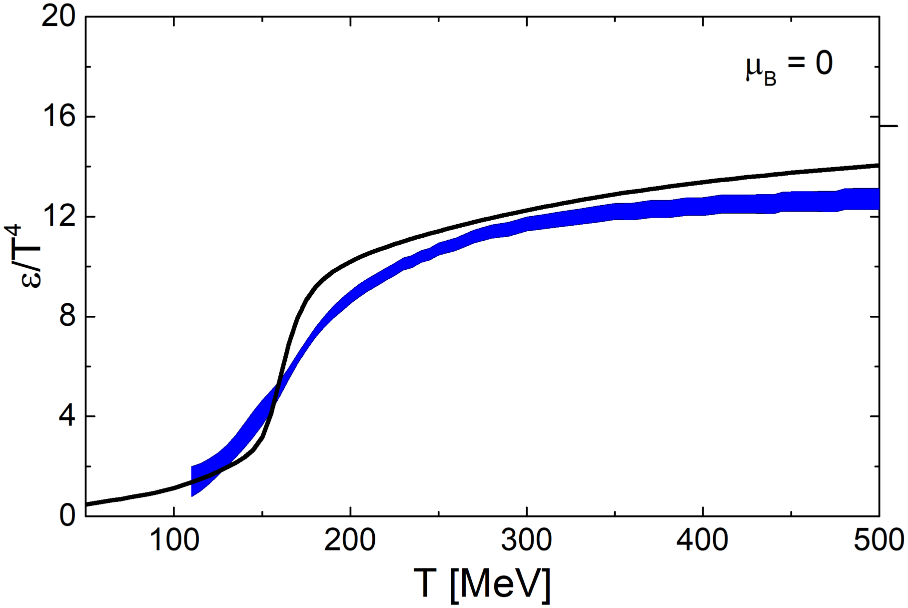
$T_H \simeq 167$ MeV

Hagedorn model: Thermodynamic functions

Pressure p/T^4



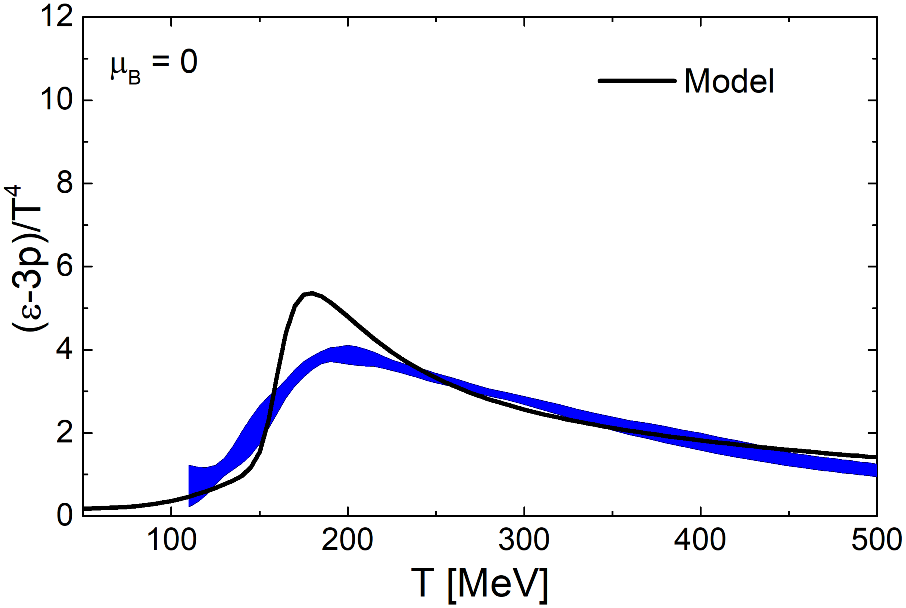
Energy density ε/T^4



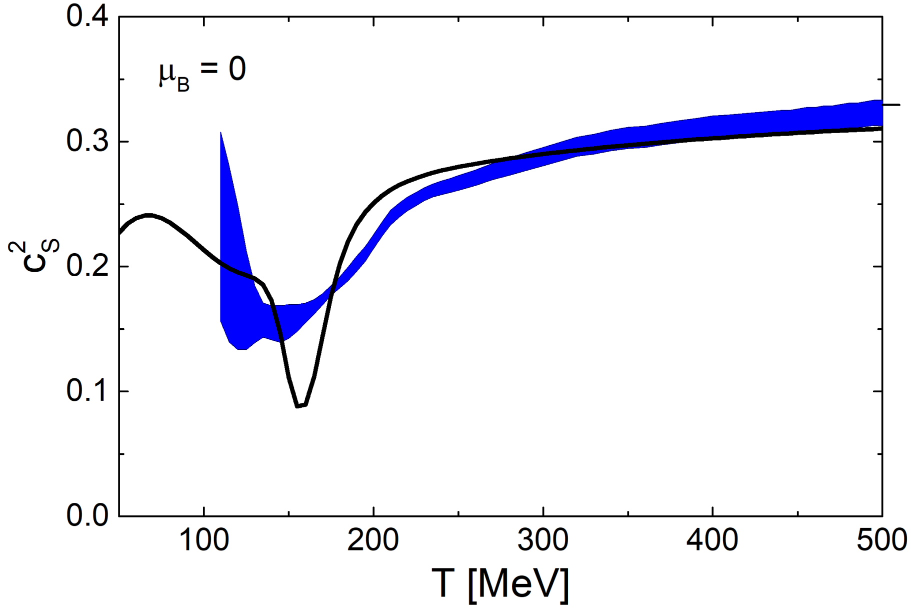
- Semi-quantitative description of lattice data
- Peak in energy density gone!

Hagedorn model: Thermodynamic functions

Trace anomaly $(\varepsilon - 3p)/T^4$

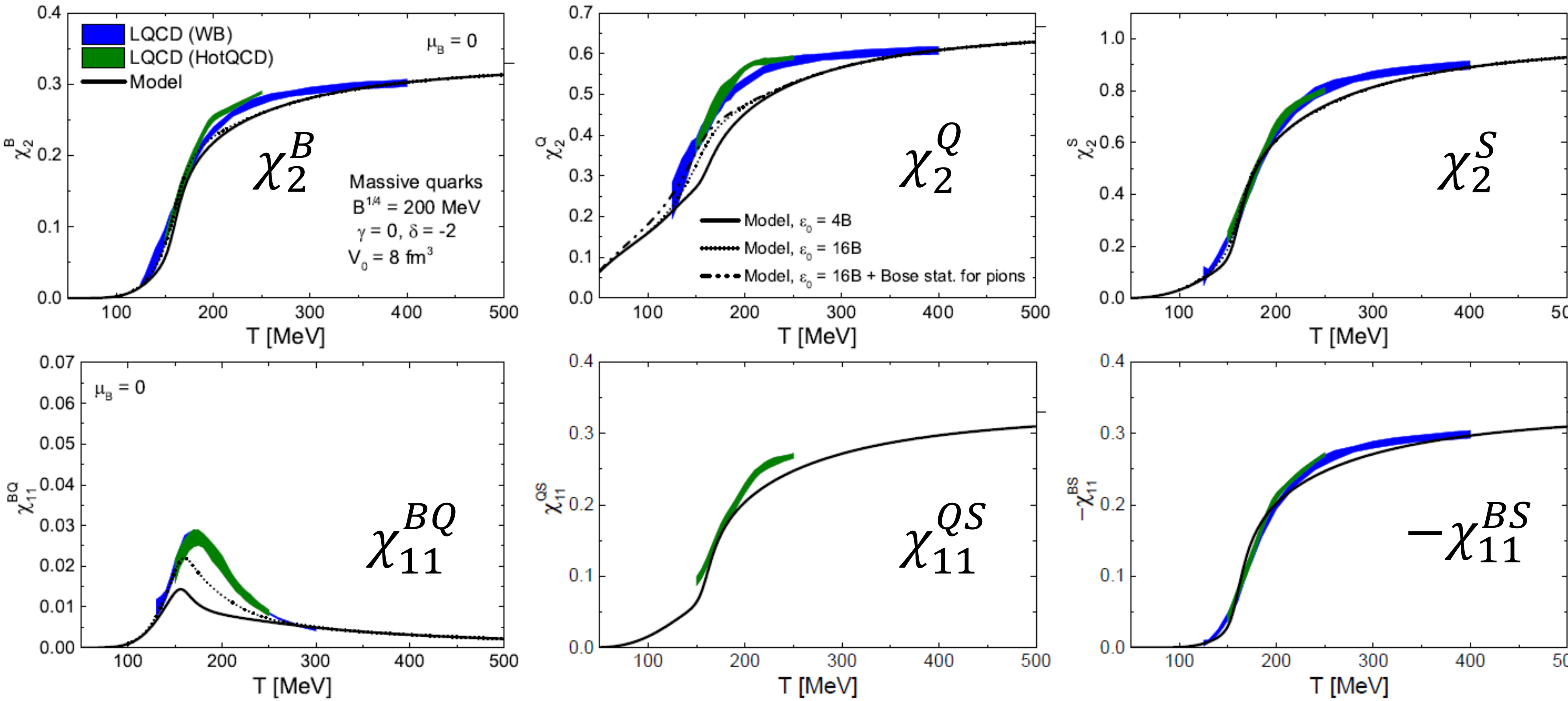


Speed of sound $c_s^2 = dp/d\varepsilon$



Hagedorn model: Susceptibilities

$$\chi^{BSQ}_{lmn} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l \, \partial(\mu_S/T)^m \, \partial(\mu_Q/T)^n}$$



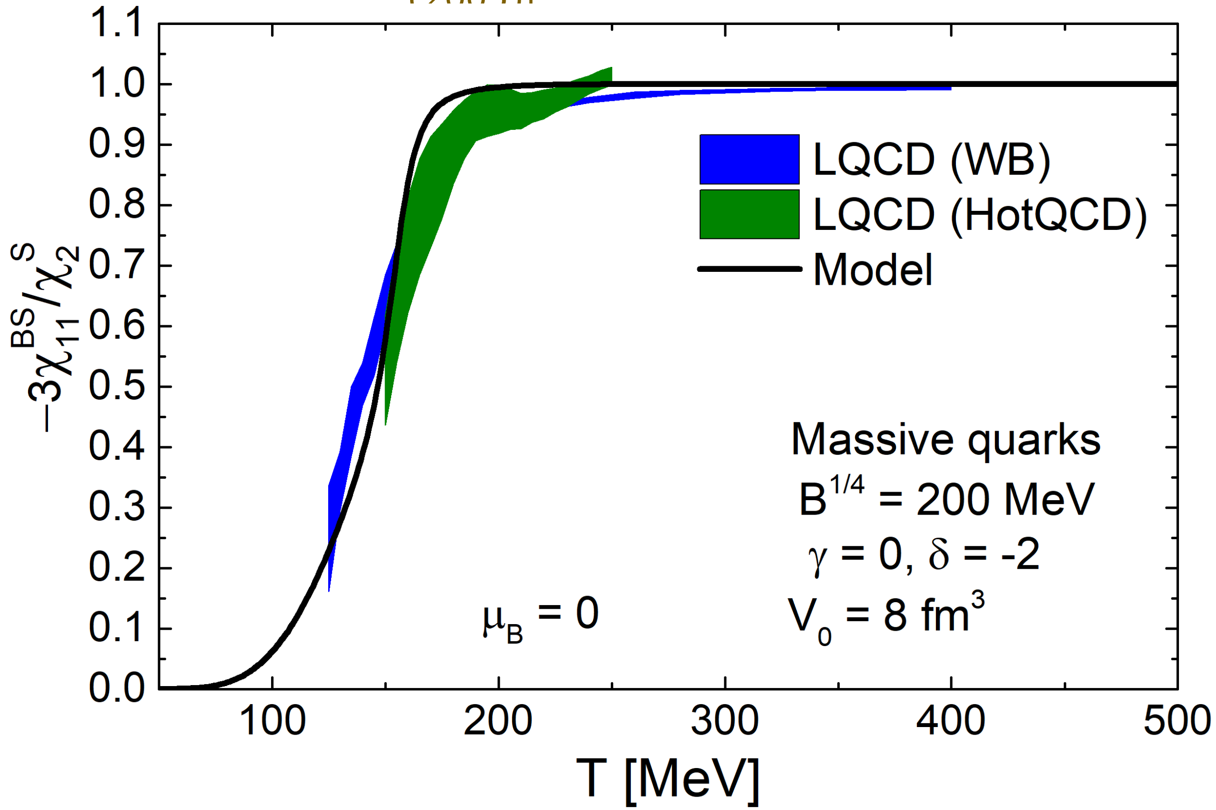
Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)

Hagedorn model: Baryon-strangeness ratio

$$C_{BS} = -\frac{3\chi_{11}^{BS}}{\chi_2^S}$$

Useful diagnostic of QCD matter

[V. Koch, Majumder, Randrup, PRL 95, 182301 (2005)]

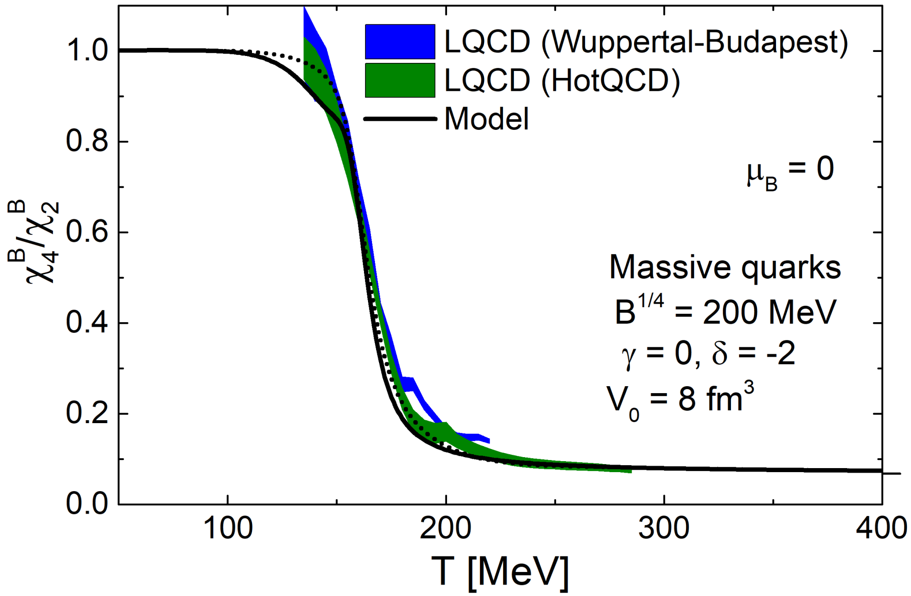


Well consistent with lattice QCD

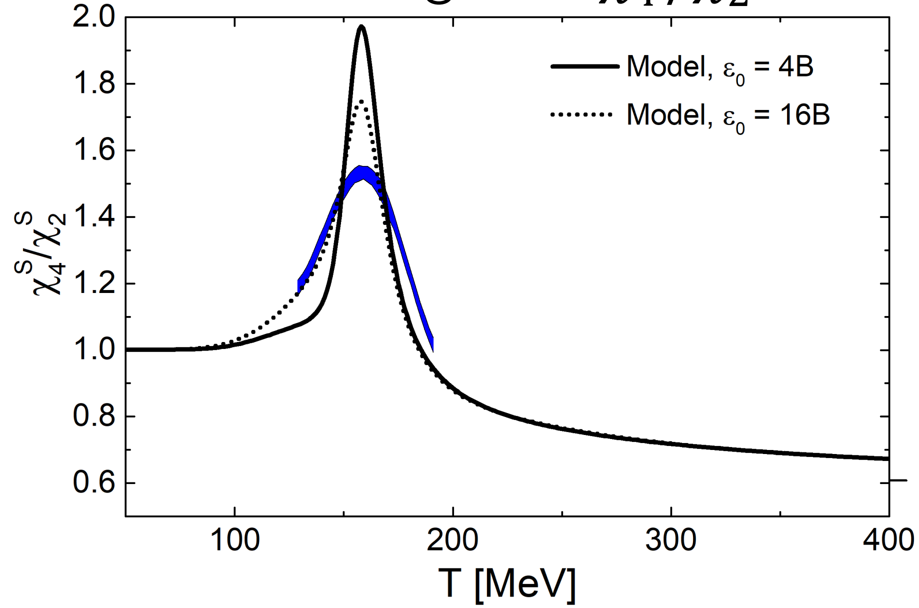
Hagedorn model: Higher-order susceptibilities

Higher-order susceptibilities are particularly sensitive probes of the parton-hadron transition and possible remnants of criticality at $\mu_B = 0$

net baryon χ_4^B/χ_2^B



net strangeness χ_4^S/χ_2^S



Lattice data from 1305.6297 & 1805.04445 (Wuppertal-Budapest), 1708.04897 (HotQCD)

- Drop of χ_4^B/χ_2^B caused by repulsive interactions which ensure crossover transition to QGP
- Peak in χ_4^S/χ_2^S is an interplay of the presence of multi-strange hyperons and repulsive interactions