Equations of state at finite baryon density

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EMMI Rapid Reaction Task Force

"Dynamics of critical fluctuations: theory – phenomenology – experiment"

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FIAS Frankfurt Institute for Advanced Studies

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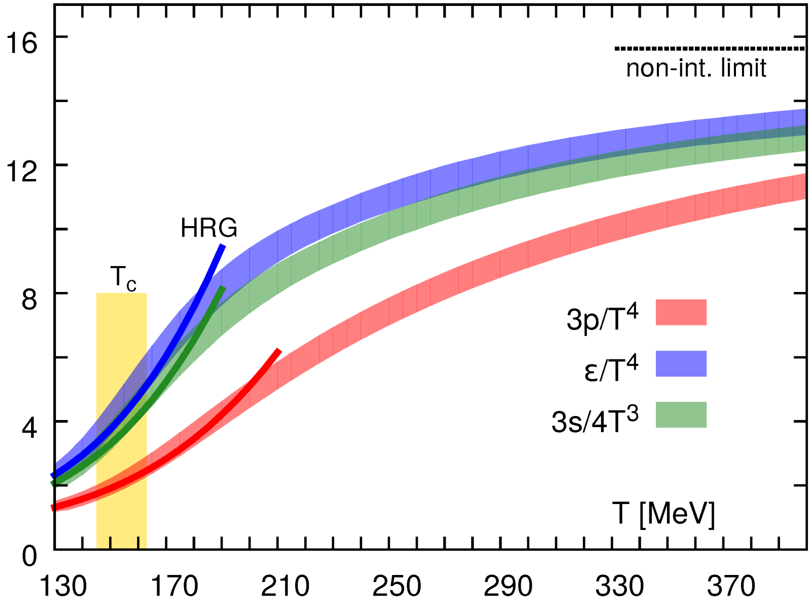
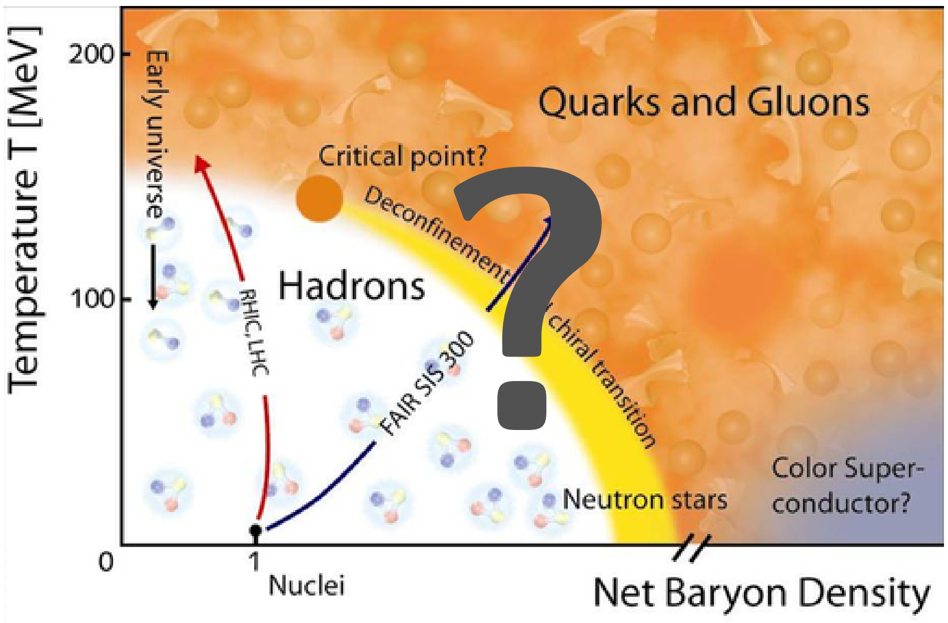
QCD phase diagram: towards finite density

$\mu_B = 0$ → ? → $T - \mu_B$ plane





- QCD EoS at $\mu_B = 0$ available from lattice QCD
- QCD EoS at finite density necessary for many applications, including hydro modeling of heavy-ion collisions at RHIC, SPS, FAIR energies
- Implementation of the QCD critical point necessary to look for its signatures

Outline

1. Taylor expansion from lattice QCD
   - Model-independent method with a limited scope (small μ_B/T)
   - State-of-the-art and estimates for radius of convergence

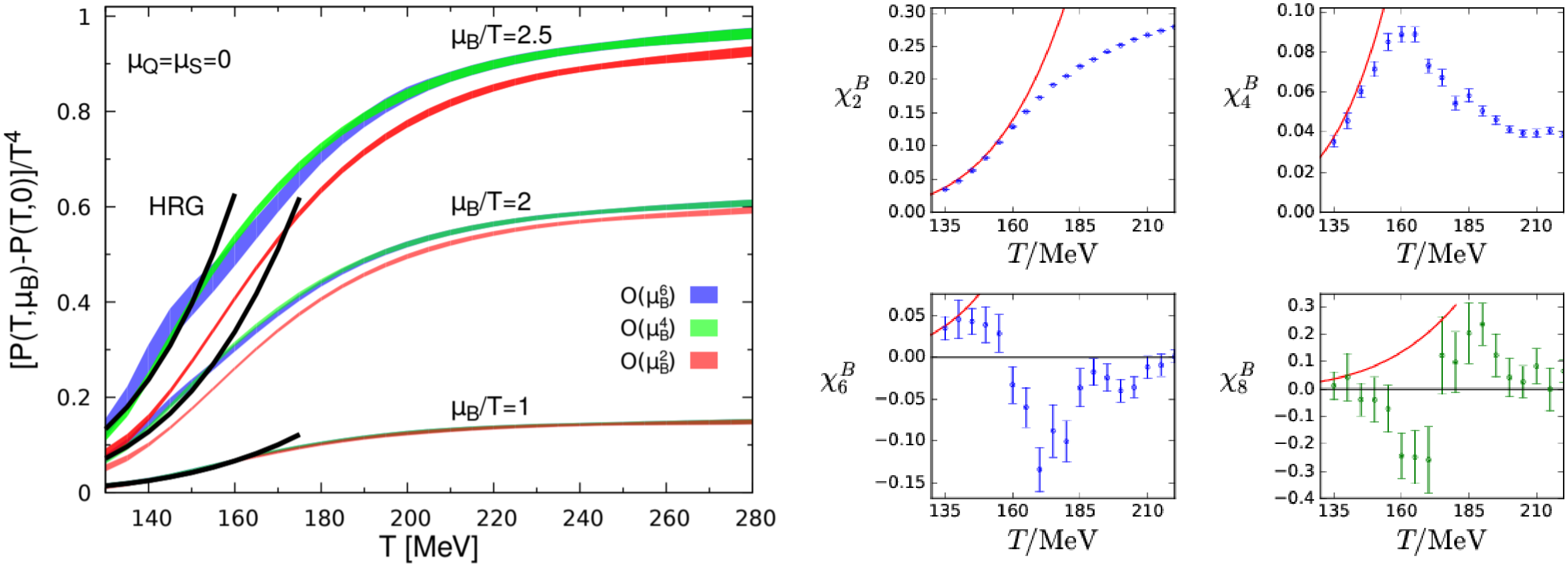
2. Lattice-based effective models
   - Cluster expansion model (CEM)
   - Hagedorn bag-like model
   - Chiral mean field model

3. Incorporating the QCD critical point
   - 3D-Ising model
   - Switching function

Finite μ_B EoS from Taylor expansion

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \frac{\chi_2^B(T,0)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T,0)}{4!}(\mu_B/T)^4 + \ldots$$

χ_k^B – cumulants of net baryon distribution, computed up to χ_8^B



[HotQCD collaboration, 1701.04325] [Wuppertal-Budapest collaboration, 1805.04445]

- Off-diagonal susceptibilities also available → incorporate conservation laws
  $n_S = 0,\ n_Q/n_B = 0.4$
- Method inherently limited to "small" μ_B/T, within convergence radius

Taylor expansion and radius of convergence

A truncated Taylor expansion only useful within the radius of convergence. Its value is a priori unknown. Any singularity in *complex* μ_B plane will limit the convergence, it does not have to be a phase transition or a critical point

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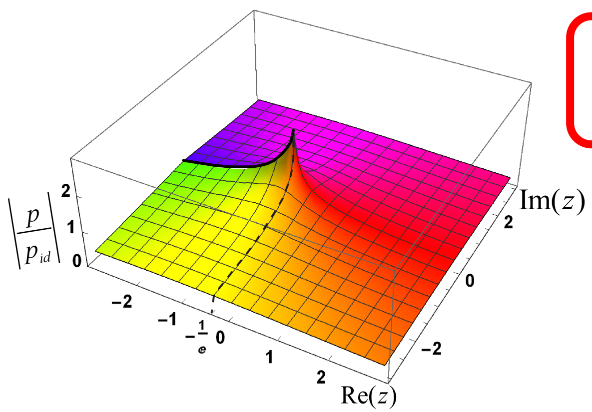
An example: HRG model with a baryonic excluded volume (EV)



Lambert W(z) function has a branch cut singularity at $z = -e^{-1}$, corresponds to a negative fugacity

[Taradiy, **V.V.,** Gorenstein, Stoecker, in preparation]

$$p(T, \mu_B) \sim W\left[b\,\phi_B(T)\,e^{\mu_B/T}\right],$$

$b \simeq 1$ fm^3

Constrained to LQCD data

[**V.V.** et al., 1708.02852]

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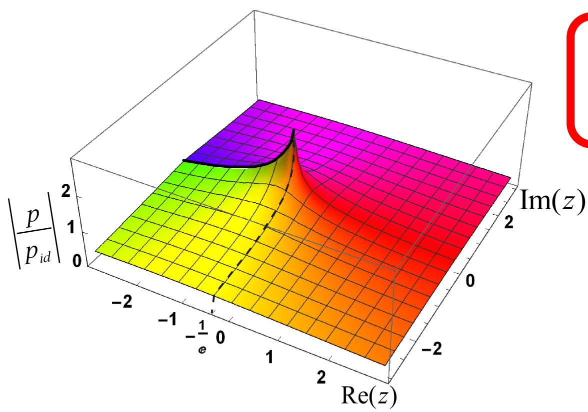

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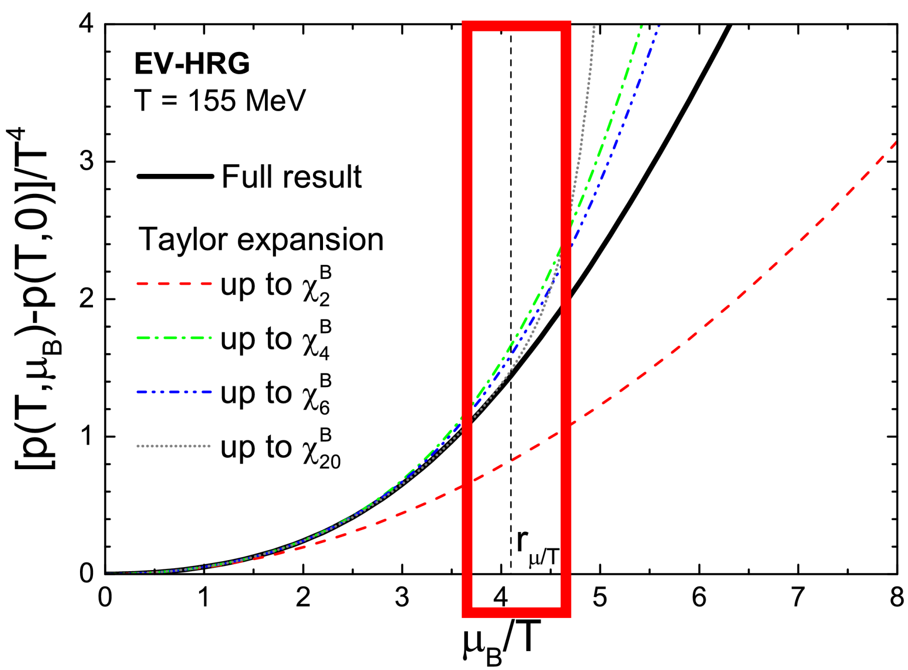
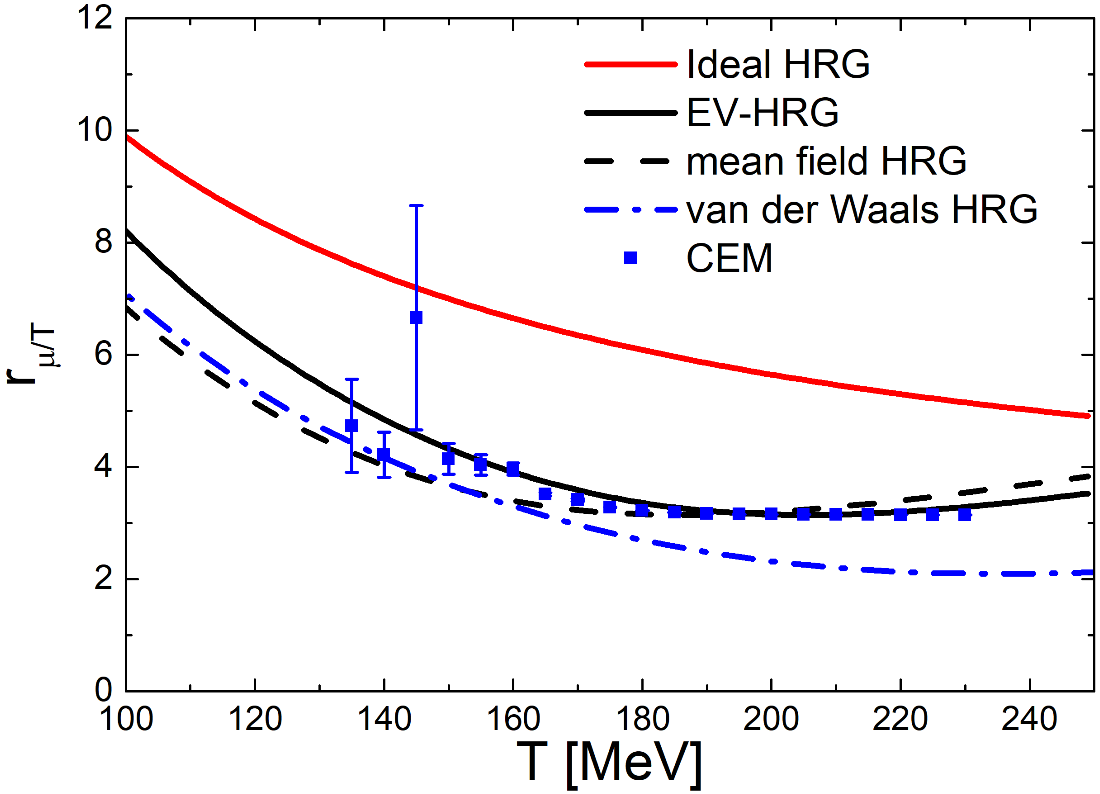


Radius of convergence from different models



Ideal HRG

Singularity in the nucleon Fermi-Dirac function

EV-HRG & mean-field HRG

[**V.V.**+, 1708.02852] [Huovinen, Petreczky, 1708.02852]

Repulsive baryonic interactions. Singularity of the Lambert W function

van der Waals HRG

[**V.V.**, Gorenstein, Stoecker, 1609.03975]

Crossover singularities connected to the nuclear matter critical point at $T \sim 20$ MeV and $\mu_B \sim 900$ MeV

see also M. Stephanov, hep-lat/0603014

Cluster Expansion Model (CEM)

[**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]

Roberge-Weiss like transition: $Im\, \frac{\mu_B}{T} = \pi$

Taylor expansion likely divergent at $\mu_B/T \geq$ 3-5,
regardless of existence of the QCD critical point

Recent Taylor-based EoS parameterizations

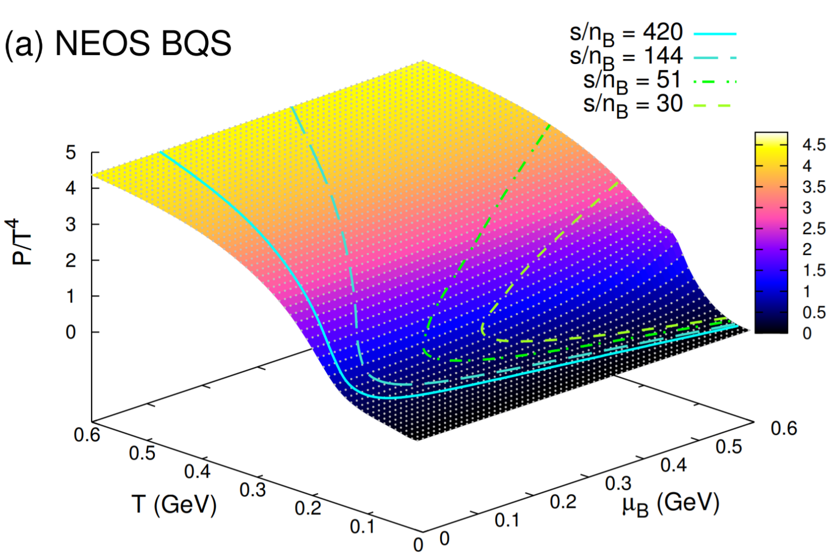
Truncated LQCD Taylor expansion

$$\frac{p}{T^4} = \sum_{i,j,k} \frac{\chi^{BQS}_{i,j,k}(T)}{i!\,j!\,k!} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

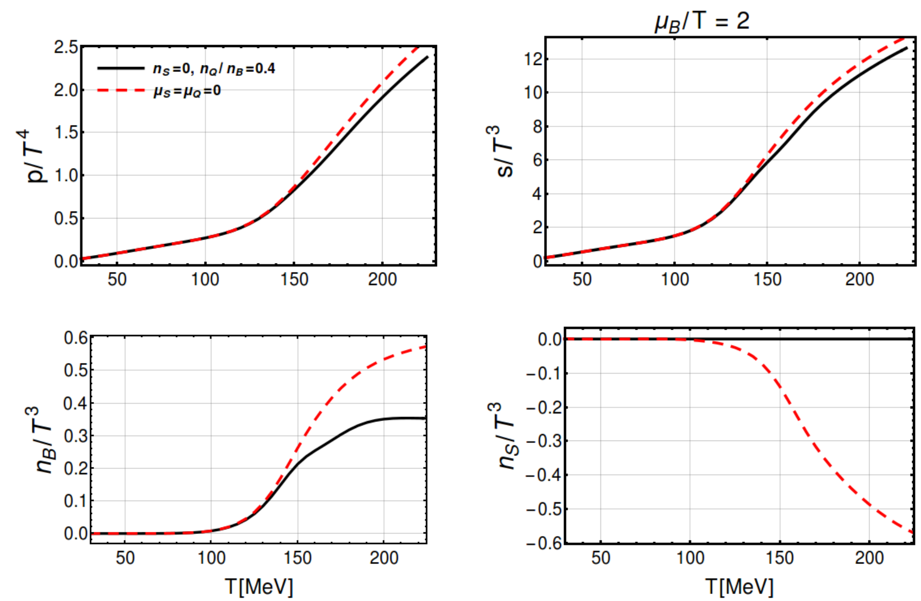
$+$

HRG model at smaller temperatures

$$\frac{p}{T^4} = \sum_{i\in \mathrm{hrg}} T\,\phi_i^{\mathrm{id}}(T)\, e^{b_i\mu_B/T}\, e^{q_i\mu_Q/T}\, e^{s_i\mu_S/T}$$



[Monnai, Schenke, Shen, 1902.05095]



[Noronha-Hostler, Parotto, Ratti, Stafford, 1902.06723]

- Includes the three conserved charges and conservation laws, no criticality
- Probably best one can do with Taylor expansion. Applications: RHIC BES

Truncated Taylor expansion and imaginary μ_B

Are we using all information available from lattice? Consider imaginary μ_B

$$\left.\frac{\rho_B}{T^3}\right|_{\mu_B = i\theta_B T} = i \sum_{k=1}^{\infty} b_k(T) \sin(k\theta_B) \qquad \Rightarrow \qquad b_k(T) = -\frac{2i}{\pi} \int_0^{\pi} \frac{\rho_B(T, i\theta_B T)}{T^3} \sin(k\,\theta_B)\, d\theta_B$$

Relativistic virial/cluster expansion

Fourier coefficients

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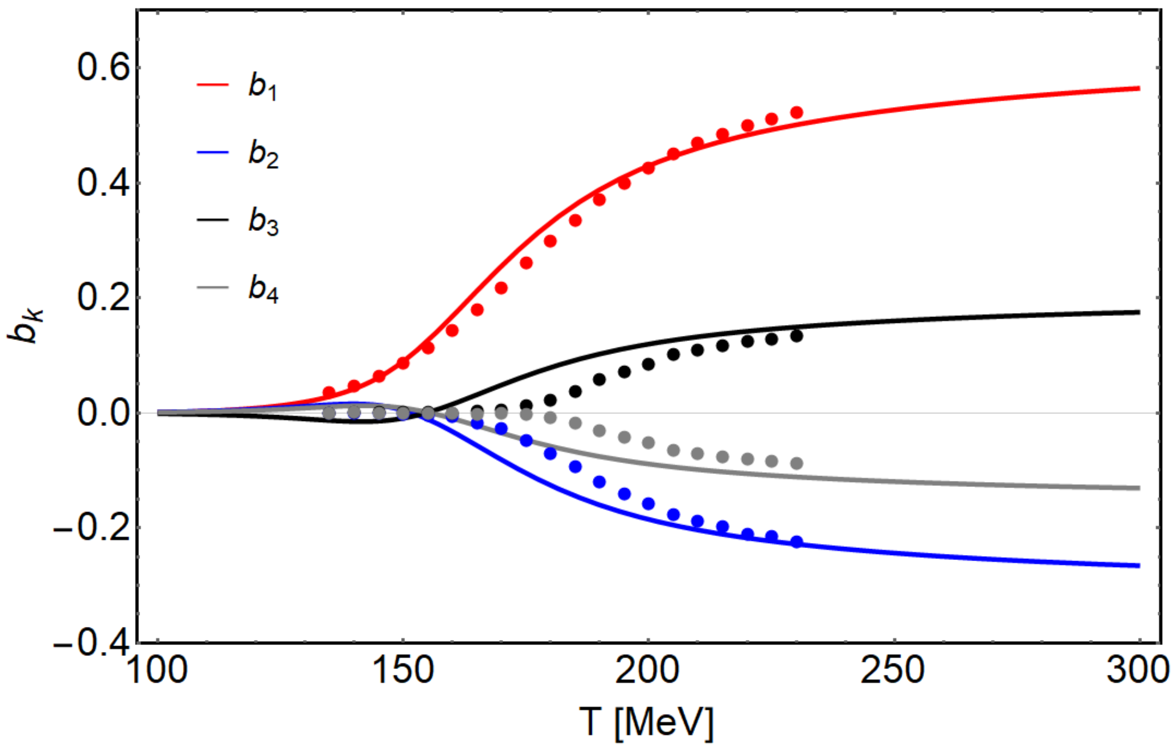
Relativistic virial/cluster expansion *Fourier coefficients*



Lines: *Taylor expansion up to χ_B^4 using lattice data, as in 1902.06723*

Symbols: *Lattice data for b_k from imaginary μ_B*

[**V.V.**, Pasztor, Fodor, Katz, Stoecker, 1708.02852]

Quite some room for improvement at T<200 MeV

Cluster Expansion Model — CEM

a model for QCD equation of state at finite baryon density
constrained to both susceptibilities and Fourier coefficients

V.V., *J. Steinheimer, O. Philipsen, H. Stoecker, Phys. Rev. D 97, 114030 (2018)*
V.V. *et al., Nucl. Phys. A 982, 859 (2019)*

Cluster Expansion Model (CEM)

Model formulation:

- Cluster expansion for baryon number density

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T)\,\sinh(k\mu_B/T)$$

- $b_1(T)$ and $b_2(T)$ are model input from lattice QCD
- All higher order coefficients are predicted: $b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$

Physical picture: Hadron gas with repulsion at moderate T, QGP-like at high T

Summed analytic form:

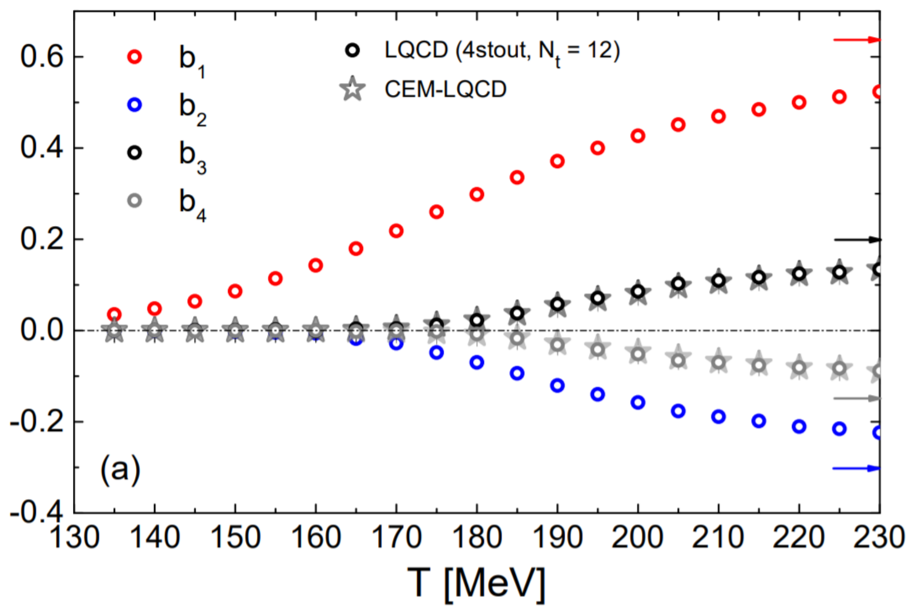
$$\frac{\rho_B(T,\mu_B)}{T^3} = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_1(x_+) - \mathrm{Li}_1(x_-)\right] + 3\left[\mathrm{Li}_3(x_+) - \mathrm{Li}_3(x_-)\right]\right\}$$

$$\hat{b}_{1,2} = \frac{b_{1,2}(T)}{b_{1,2}^{\mathrm{SB}}}, \quad x_\pm = -\frac{\hat{b}_2}{\hat{b}_1} e^{\pm\mu_B/T}, \quad \mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$$

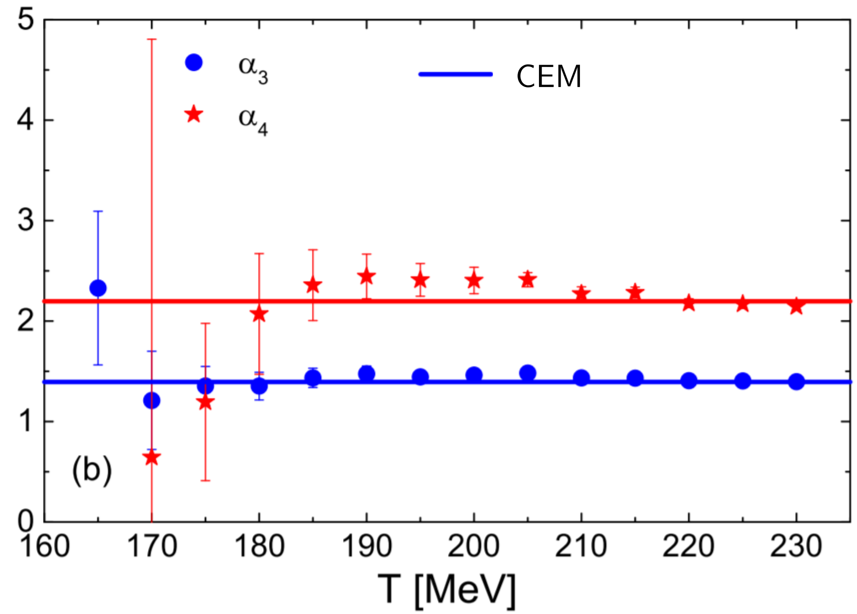
Regular behavior at real μ_B → *no-critical-point scenario*

CEM: Fourier coefficients

$$b_k(T)$$



$$\alpha_k(T) \equiv b_k(T)\,\frac{[b_1(T)]^{k-2}}{[b_2(T)]^{k-1}}$$

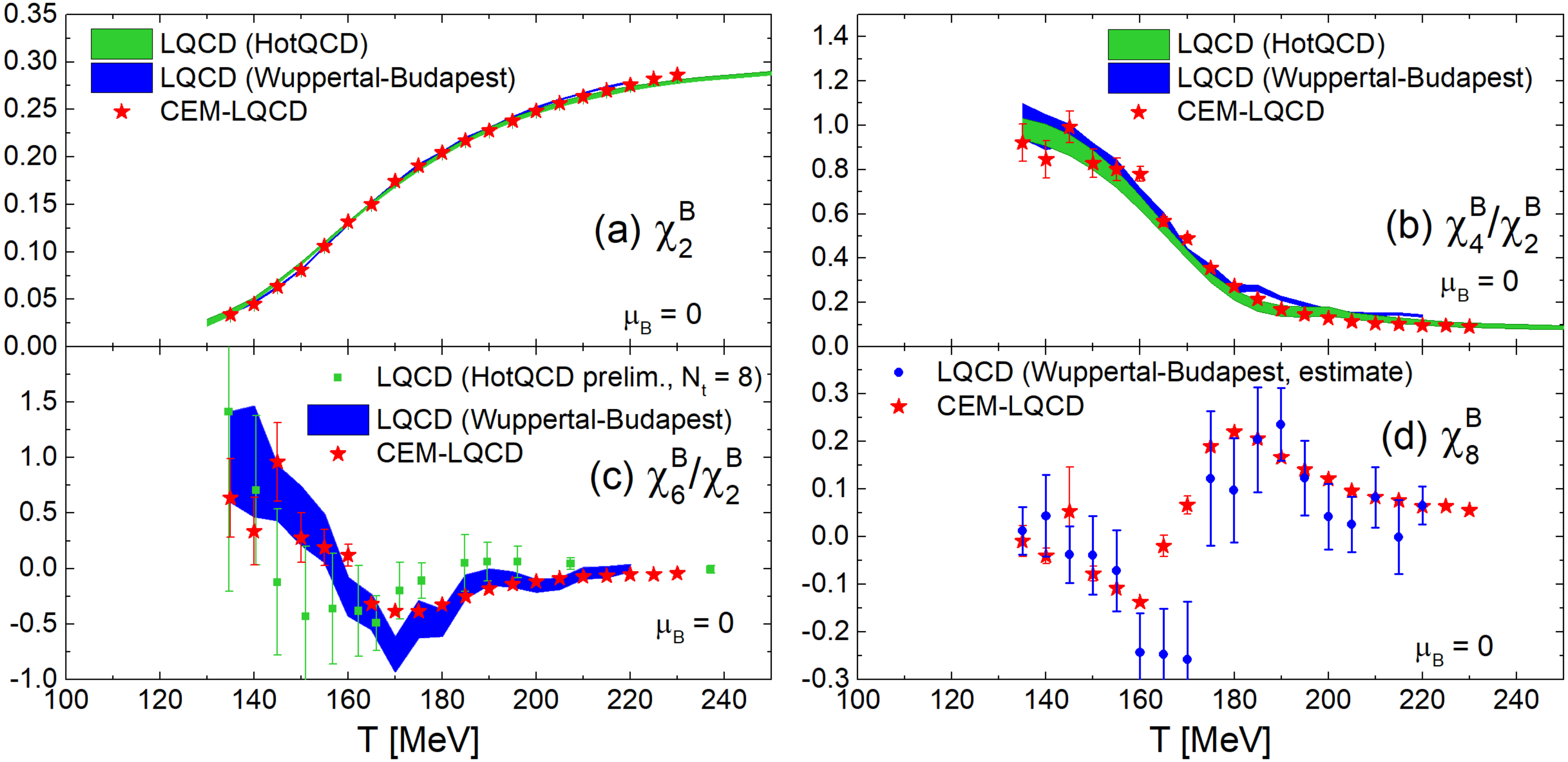


CEM: $b_1(T)$ and $b_2(T)$ as input, consistent description of $b_3(T)$ and $b_4(T)$

Lattice data on $b_{3,4}(T)$ inconclusive at $T \leq 170$ MeV

CEM: Baryon number susceptibilities

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2\text{-}k}(x_+) + (-1)^k\,\mathrm{Li}_{2\text{-}k}(x_-)\right] + 3\left[\mathrm{Li}_{4\text{-}k}(x_+) + (-1)^k\,\mathrm{Li}_{4\text{-}k}(x_-)\right]\right\}$$

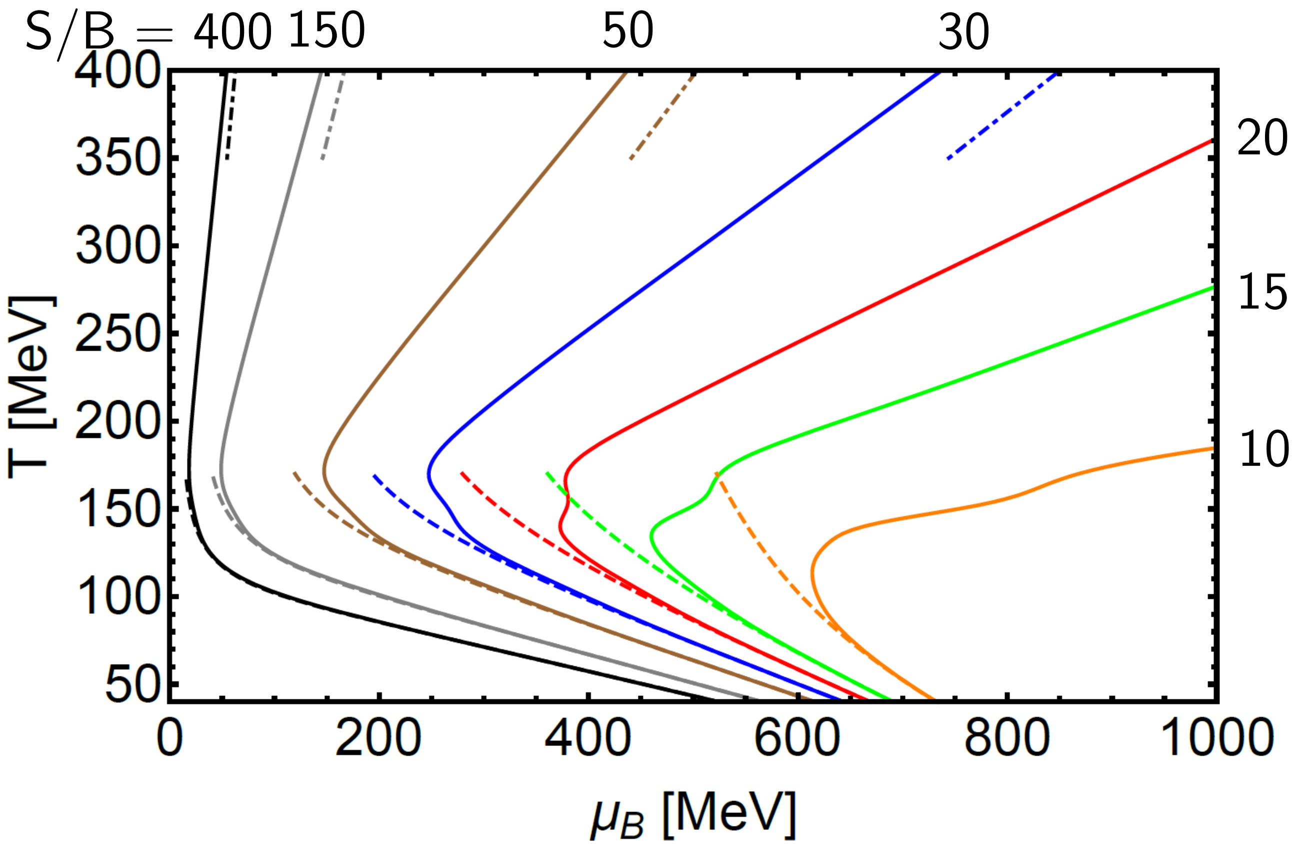


Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)

CEM: Equation of state

$$\frac{p(T,\mu_B)}{T^4} = p_0(T) - \frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2[\mathrm{Li}_2(x_+) + \mathrm{Li}_2(x_-)] + 3[\mathrm{Li}_4(x_+) + \mathrm{Li}_4(x_-)]\right\}$$

Input: $p_0(T),\ b_{1,2}(T)$ $\leftarrow$ parametrized LQCD + excluded volume HRG



Hagedorn (bag-like) resonance gas model with repulsive interactions

exactly solvable model with a (phase) transition between hadronic matter and QGP

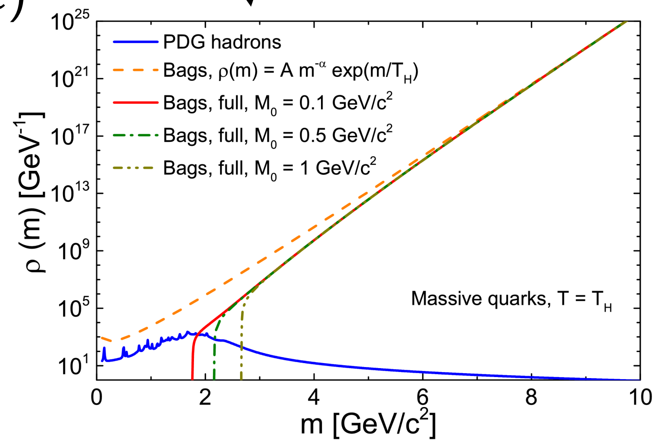
[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; Ferroni, Koch, PRC '09]

Here the model equation of state is constrained to lattice QCD

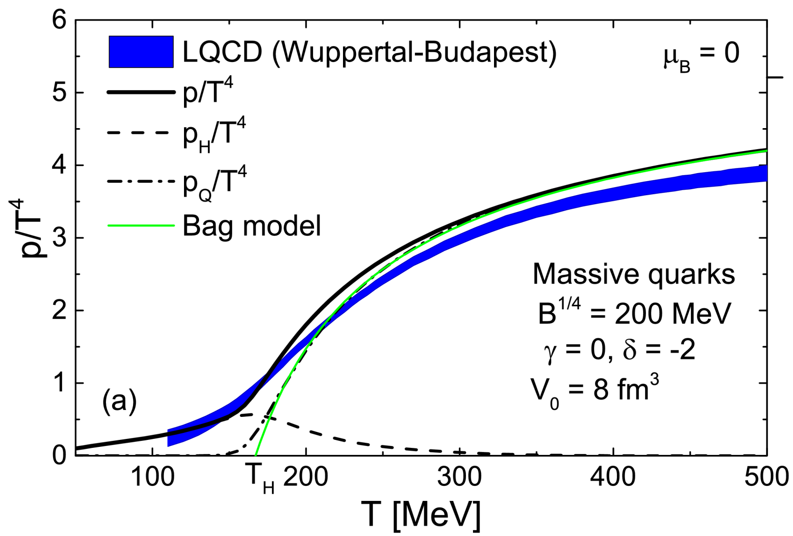
V.V., *M.I. Gorenstein, C. Greiner, H. Stoecker, arxiv:1811.05737, Phys. Rev. C in press*

Hagedorn bag-like model: formulation

- HRG + quark-gluon bags $\rho_Q(m, v) = C\, v^\gamma\, (m - Bv)^\delta \exp\left\{\frac{4}{3}[\sigma_Q]^{1/4}\, v^{1/4}\, (m - Bv)^{3/4}\right\}$
- Non-overlapping particles (excluded volume correction) $V \to V - bN$
- Isobaric (pressure) ensemble $(T, V, \mu) \to (T, s, \mu)$
- Massive (thermal) quarks and gluons



Resulting picture: transition (crossover, 1st order, 2nd order, etc.) between HRG and MIT bag model EoS, within single partition function

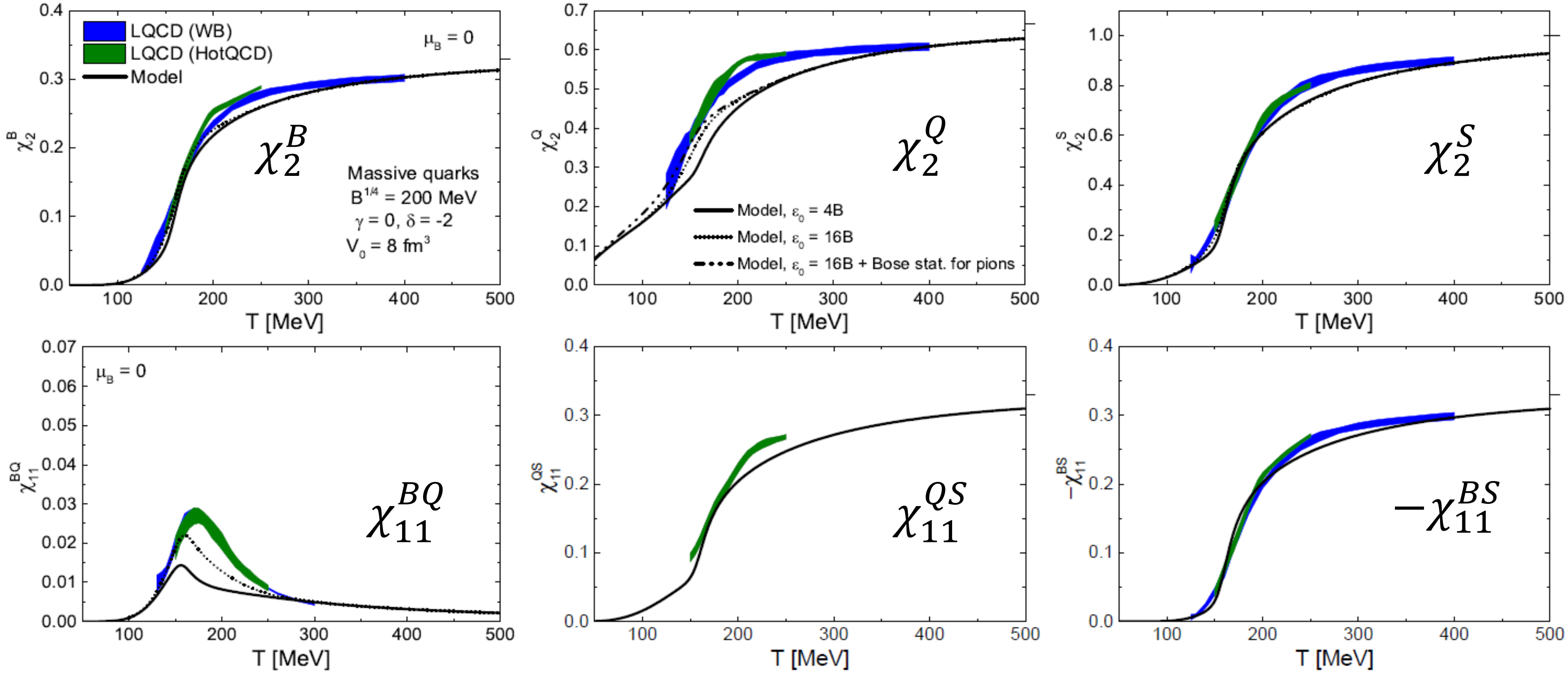


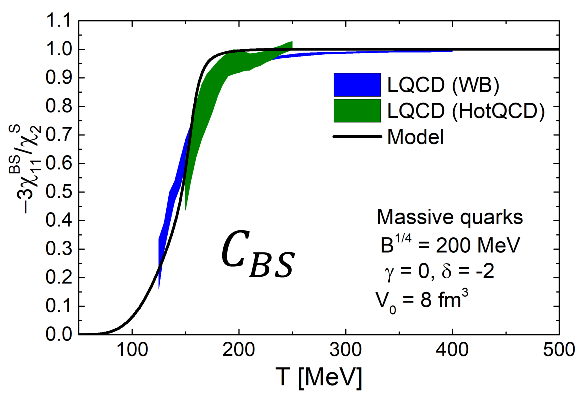
"Crossover" parameter set

$$\gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3$$
$$m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV}$$
$$m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV}$$

$T_H \simeq 167$ MeV

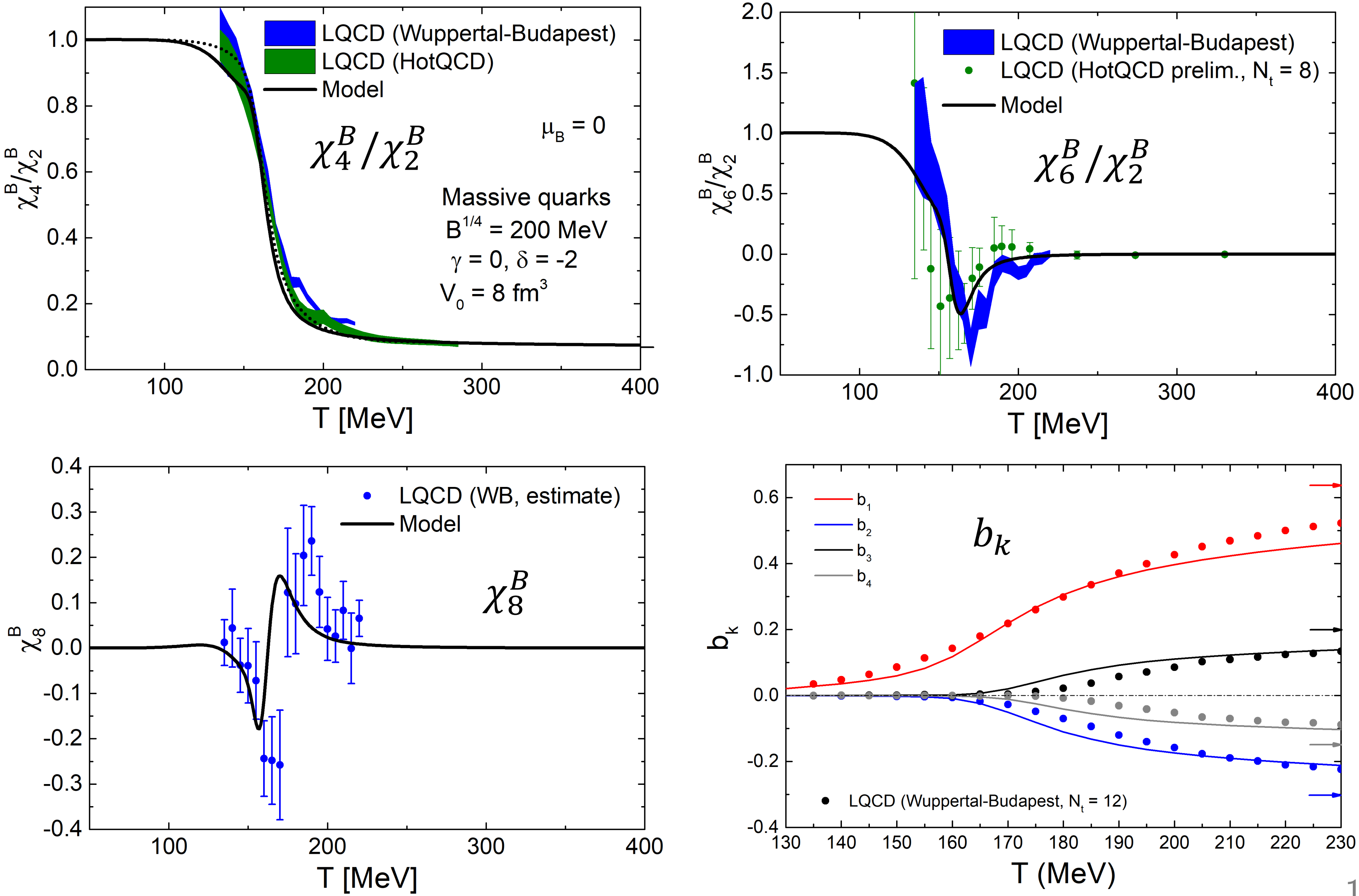
Hagedorn model: Susceptibilities



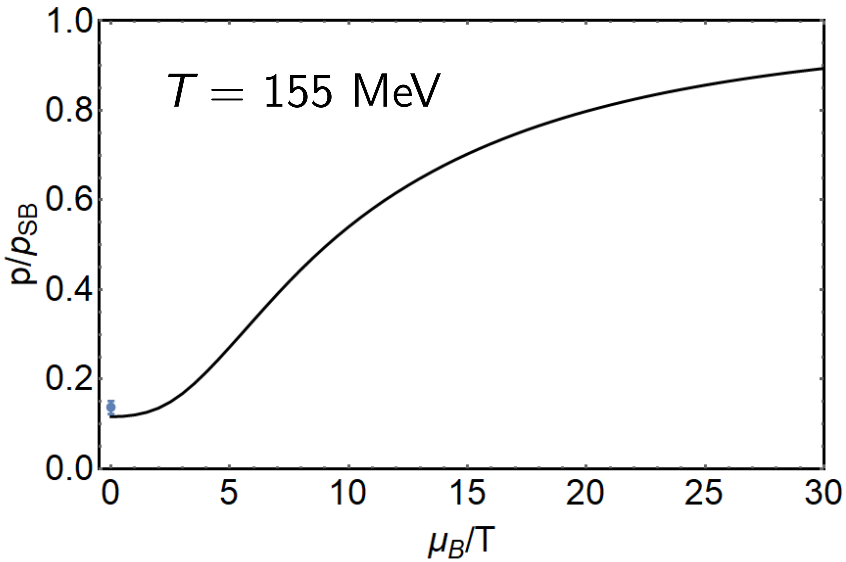


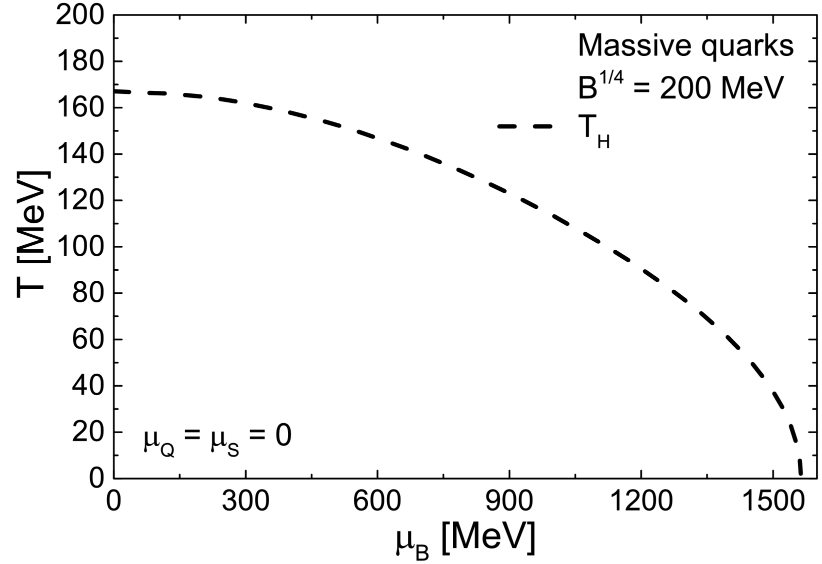
Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)

Hagedorn model: Susceptibilities and Fourier



Hagedorn model: Finite baryon density





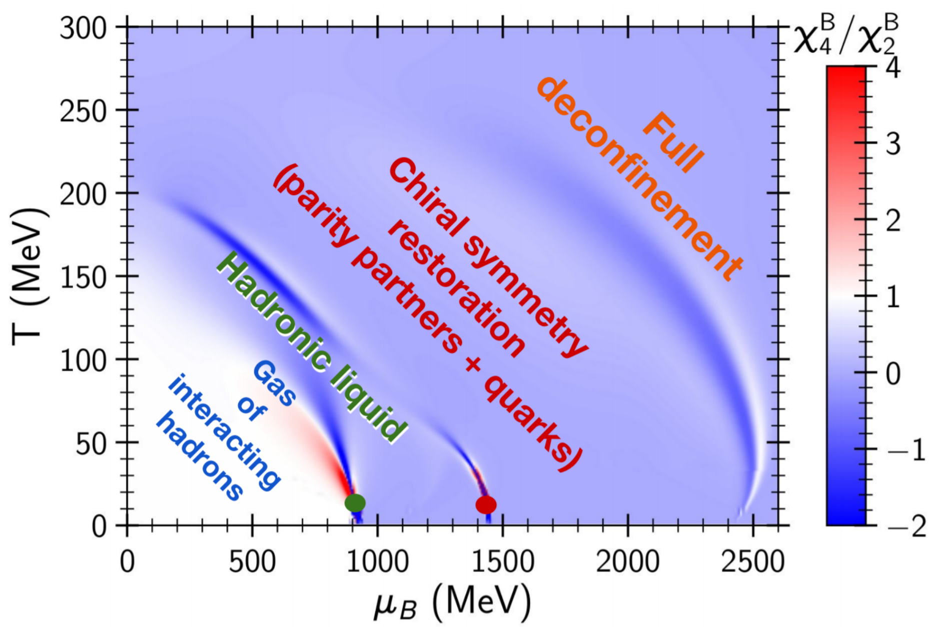
- Transition to a QGP-like phase in both the T and μ_B directions
- Essentially a built-in "switching" function between HRG and QGP, thermodynamically consistent by construction (single partition function)
- Critical point/phase transition at finite μ_B can be incorporated through μ_B-dependence of γ and δ exponents in bag spectrum

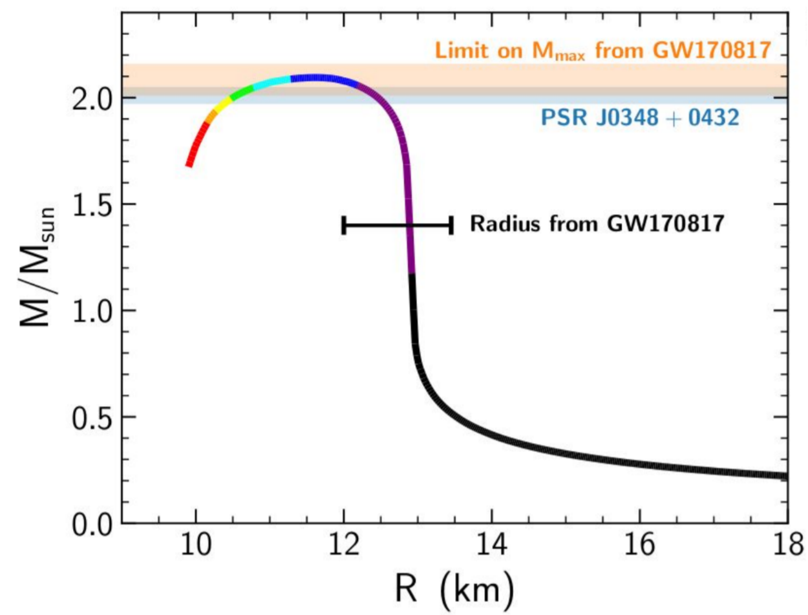
see Gorenstein, Gazdzicki, Greiner, Phys. Rev. C (2005)

SU(3) parity-doublet quark-hadron chiral model

*Motornenko, **V.V.**, Steinheimer, Schramm, Stoecker, arxiv:1809.02000 & in preparation*

- Baryons interacting through mean fields + parity doubling + excluded volume
- Quarks in a PNJL-like approach
- Constrained to lattice data at $\mu_B = 0$ and empirical nuclear matter properties





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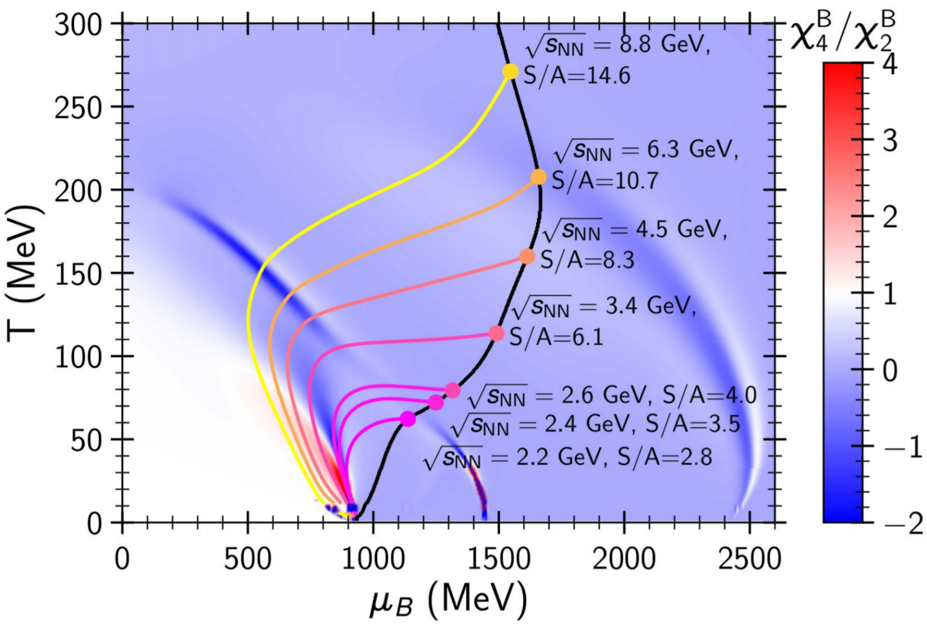
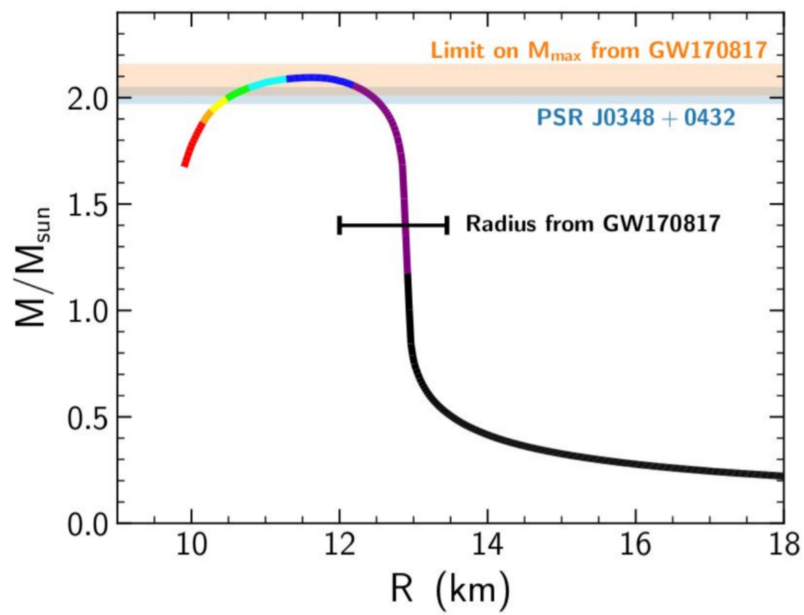




Equation of state for nuclear collisions and neutron stars (mergers)

see also Marczenko, Blaschke, Redlich, Sasaki, 1805.06886

Incorporating a critical point at finite density into the equation of state

currently no indications for the location of QCD critical point from lattice data, "small" $\mu_B/T \leq$ 2-3 disfavored

Strategy: incorporate a critical point in a way consistent with the available lattice data and make predictions testable in heavy-ion collisions/lattice QCD dependent on CP location

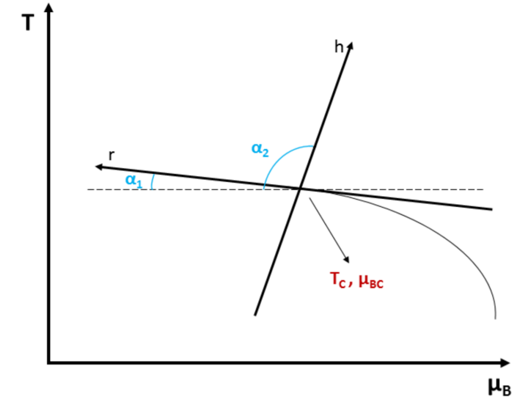
EoS with a CP from the 3D Ising Model

P. Parotto et al., arxiv:1805.05249

Introduce a CP through the scaling EoS of the 3D Ising Model and map Ising variables to T and μ_B in QCD

$$M = M_0 R^\beta \theta \ , \quad h = h_0 R^{\beta\delta} \tilde{h}(\theta) \ , \quad r = R(1-\theta^2) \ .$$

$$\frac{T - T_C}{T_C} = w\,(r\rho\,\sin\alpha_1 + h\,\sin\alpha_2) \ , \quad \frac{\mu_B - \mu_{BC}}{T_C} = w\,(-r\rho\,\cos\alpha_1 - h\,\cos\alpha_2)$$



$$p(T,\mu_B) = p^{\mathrm{non-Ising}}(T,\mu_B) + p^{\mathrm{Ising}}(T,\mu_B)$$

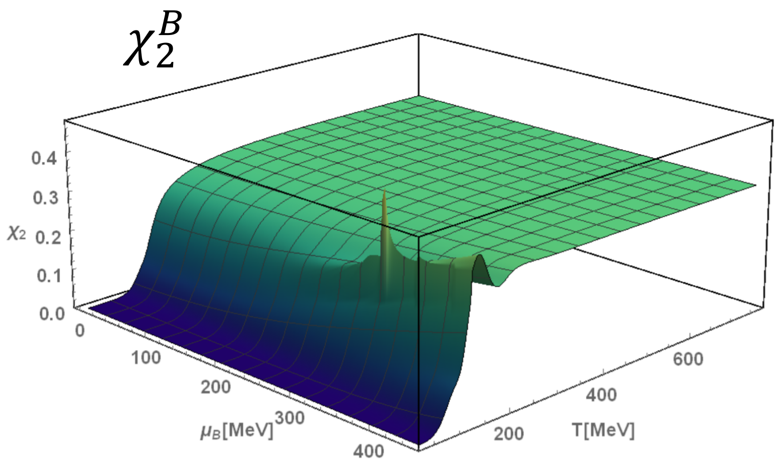
regular **critical**

made to match LQCD up to $O(\mu_B^4)$

critical point location can be varied

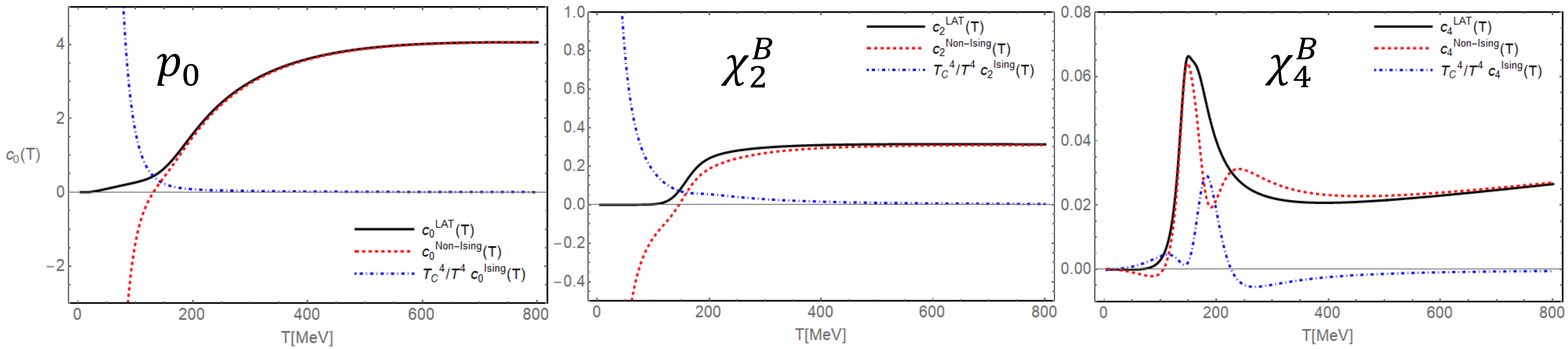
Example: $T_C = 143$ MeV, $\mu_{BC} = 350$ MeV

on the chiral pseudocritical line



EoS with a CP from the 3D Ising Model

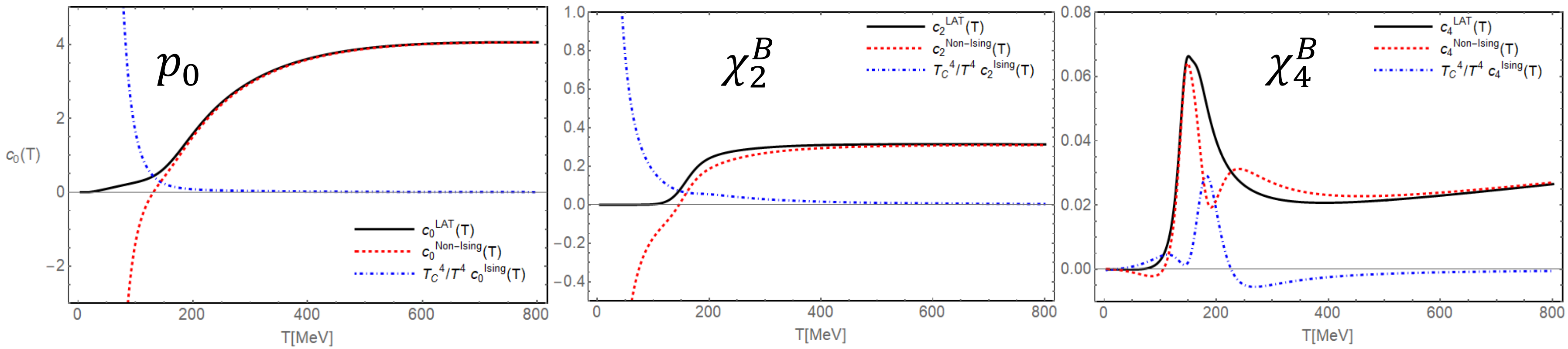
P. Parotto et al., arxiv:1805.05249



As strong temperature dependence of "regular" part as that of "singular"

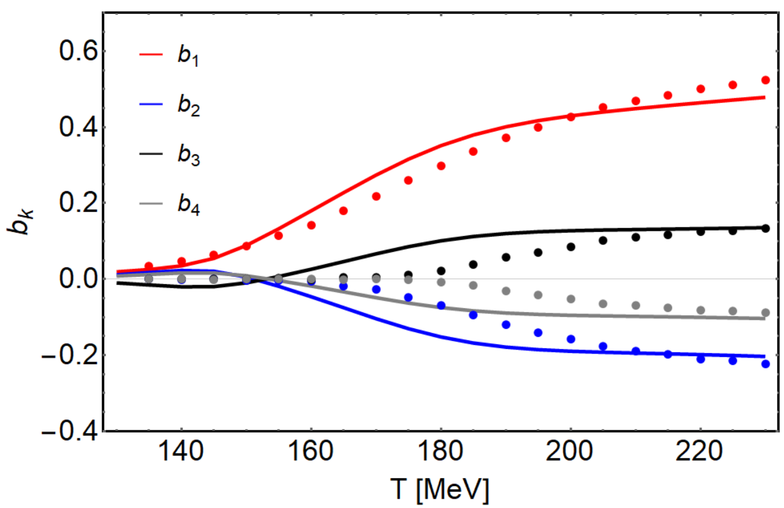
EoS with a CP from the 3D Ising Model

P. Parotto et al., arxiv:1805.05249



As strong temperature dependence of "regular" part as that of "singular"

Fourier coefficients



- No description of Fourier coefficients
- Can this be used to rule out certain regions for CP?
- Or method can be modified to incorporate lattice data for b_k?

EoS with a CP using the switching function

Plumberg, Welle, Kapusta, arxiv:1812.01684

$$p(T,\mu) = S(T,\mu)\, p_{\mathrm{QGP}}(T,\mu) + [1 - S(T,\mu)]\, p_{\mathrm{HRG}}(T,\mu)$$

hard-thermal loop — *excluded volume HRG*

CP through a judicious choice of the switching function $S(T,\mu) \sim \arg(x+iy)$

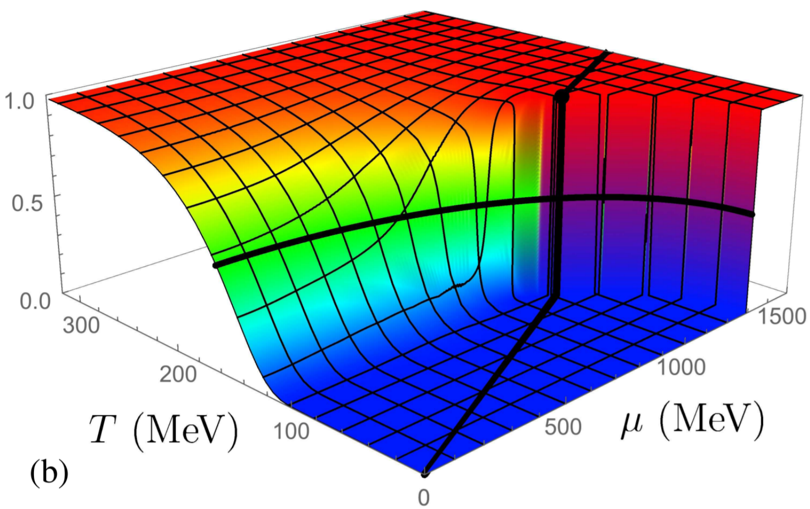
$$S(T,\mu,\psi_c,r) = \frac{1}{2} + \frac{1}{\pi}\arg\left(\eta_1(m,t,\psi_c) + i\eta_2(m,t,r)\right)$$

$$\eta_1(\mu,T,\psi_c) \equiv \frac{1}{2}\left[1 + \tanh\left(\frac{a\left(b - \left|\frac{\psi}{\psi_c}\right|\right)}{\left|\frac{\psi}{\psi_c}\right|\left(1 - \left|\frac{\psi}{\psi_c}\right|\right)}\right)\right]$$

$$\eta_2(\mu,T,r) \equiv \tan\left[\frac{\pi}{2^\theta} - \frac{\pi}{2}\right],$$

$$\theta(\mu,T,r) = \left(\frac{T^2+\mu^2}{R^2(\psi)}\right)^{-r/2},$$

$$\psi = \arctan(\mu/T),$$



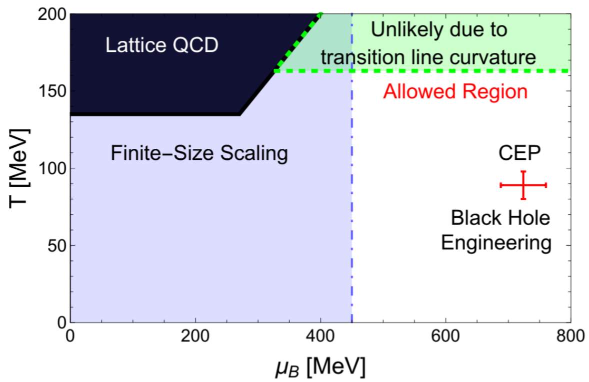
- CP location can be varied
- Currently restricted to mean-field universality class

for a crossover EoS using this method see M. Albright et al., 1404.7540 & 1506.03408

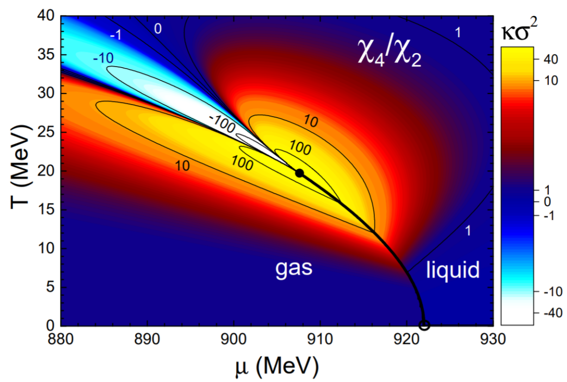
EoS with a criticality: other approaches

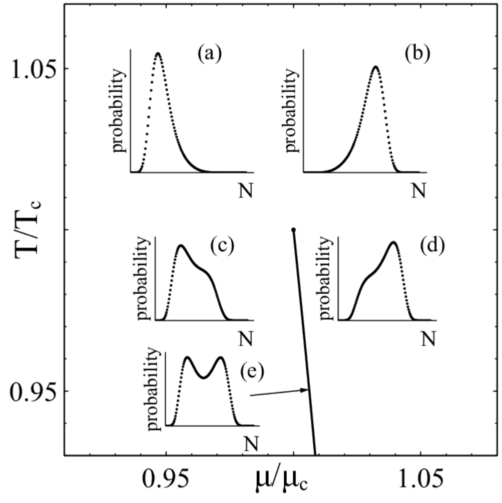
1. Holographic gauge/gravity correspondence
   - Einstein-Maxwell dilaton model
   - Constrained to LQCD data on s/T^3 and χ_2^B
   - Critical point at T ~ 90 MeV and μ_B ~ 720 MeV

Critelli et al., 1706.00455;
see also Knaute et al., 1702.06731; Li et al., 1706.02238



2. Parameterize phase transition with a van der Waals-like form
   - Originally, e.g. the vdW-HRG model with a nuclear matter critical point
     ***V.V.**, Gorenstein, Stoecker, 1609.03975*
   - Parameters can be tuned put a CP in a specific location
     see e.g. Samanta, Mohanty, 1709.04446; Bzdak et al. 1804.04463





Summary

- Steady progress from lattice QCD on Taylor expansion coefficients of QCD EoS at finite μ_B, μ_Q, μ_S for "small" densities.

- Reasonable (crossover) equation of state at moderate μ_B can be obtained in effective models constrained to all available lattice data.
  Examples: Cluster Expansion Model, Hagedorn bag-like model, Chiral mean-field model, etc.

- Approaches to incorporate a critical point in a given location and predict its signatures in heavy-ion collisions/new lattice data recently has been developed (3D-Ising, switching function, van der Waals). Still need to be matched to the available lattice constraints.

Summary

- Steady progress from lattice QCD on Taylor expansion coefficients of QCD EoS at finite μ_B, μ_Q, μ_S for "small" densities.

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Thanks for your attention!

Backup slides

Cluster expansion in fugacities

Expand in fugacity $\lambda_B = e^{\mu_B/T}$ instead of μ_B/T – a relativistic analogue of Mayer's cluster expansion:

$$\frac{p(T,\mu_B)}{T^4} = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_{|k|}(T)\, e^{k\mu_B/T} = \frac{p_0(T)}{2} + \sum_{k=1}^{\infty} p_k(T)\cosh(k\mu_B/T)$$

Net baryon density: $\frac{\rho_B(T,\mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T)\sinh(k\mu_B/T), \quad b_k \equiv k p_k$

Analytic continuation to imaginary μ_B yields trigonometric Fourier series

$$\frac{\rho_B(T,\mathrm{i}\tilde{\mu}_B)}{T^3} = \mathrm{i}\sum_{k=1}^{\infty} b_k(T)\,\sin\left(\frac{k\,\tilde{\mu}_B}{T}\right)$$

with Fourier coefficients $b_k(T) = \frac{2}{\pi\, T^4}\int_0^{\pi T} d\tilde{\mu}_B\,[\mathrm{Im}\,\rho_B(T,i\tilde{\mu}_B)]\,\sin(k\,\tilde{\mu}_B/T)$

Four leading coefficients b_k computed in LQCD at the physical point

[**V.V.**, A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852]

Why cluster expansion is interesting?

Convergence properties of cluster expansion determined by singularities of thermodynamic potential in complex fugacity plane → encoded in the asymptotic behavior of the Fourier coefficients b_k

Examples:

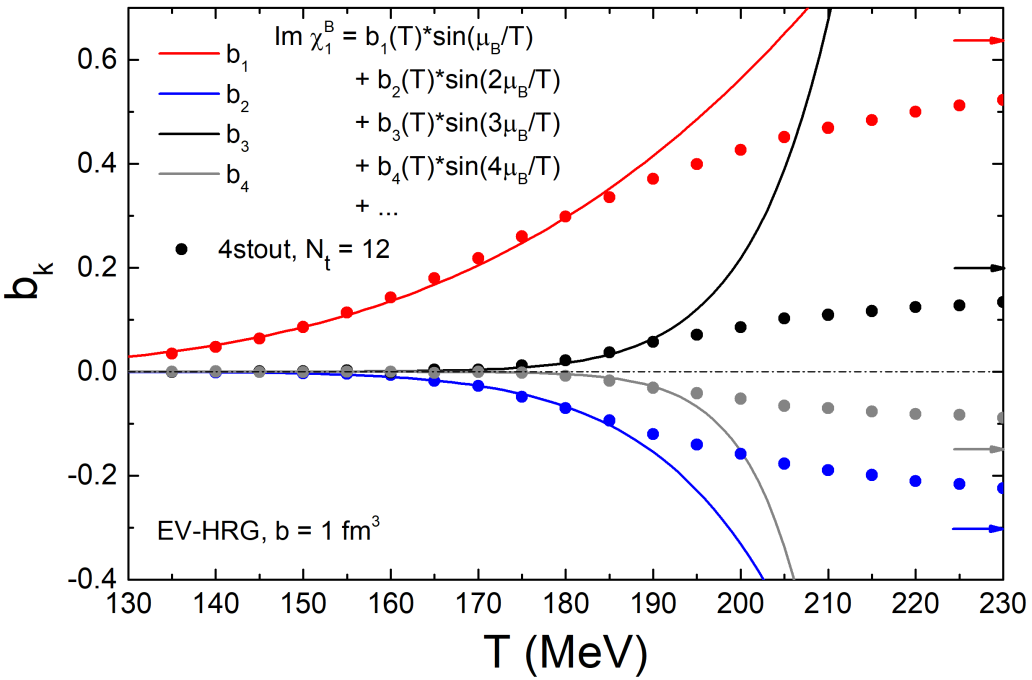
- ideal quantum gas $\quad b_k \sim (\pm 1)^{k-1} \dfrac{e^{-km/T}}{k^{3/2}}$ *Bose-Einstein condensation*
- cluster expansion model $\quad b_k \sim (-1)^{k-1} \dfrac{|\lambda_{\rm br}|^{-k}}{k}$ $|\lambda_{br}| = 1 \to$ *Roberge-Weiss transition at imaginary* μ_B
  [**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]
- excluded volume model $\quad b_k \sim (-1)^{k-1} \dfrac{|\lambda_{\rm br}|^{-k}}{k^{1/2}}$ *No phase transition, but a singularity at a negative* λ
  [Taradiy, **V.V.**, Gorenstein, Stoecker, in preparation]
- chiral crossover $\quad b_k \sim \dfrac{e^{-k\tilde{\mu}_c}}{k^{2-\alpha}} \sin(k\theta_c + \theta_0)$ *Remnants of chiral criticality at* $\mu_B = 0$
  [Almasi, Friman, Morita, Redlich, 1902.05457]

This work: signatures of a CP and a phase transition at finite density

HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) $V \to V - bN$

[Hagedorn, Rafelski, '80; Dixit, Karsch, Satz, '81; Cleymans et al., '86; Rischke et al., Z. Phys. C '91]



HRG with baryonic EV:

$$p_B(T, \mu_B) = p_B^{\mathrm{id}}(T, \mu_B - b p_B)$$

$$b_k^{ev}(T) = (-1)^{k-1} \frac{2\,k^k}{k!} (b\,T^3)^{k-1} \left[\frac{\phi_B(T)}{T^3}\right]^k$$

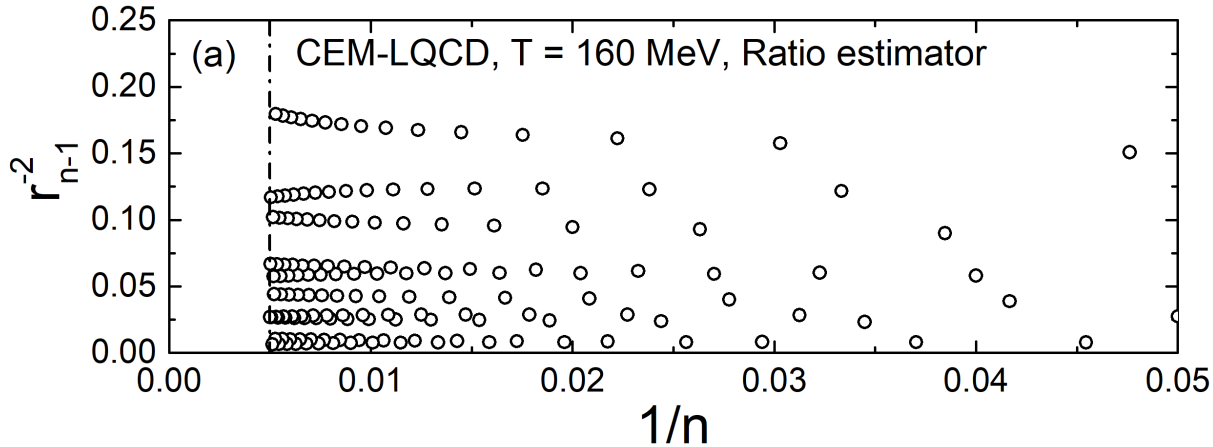
V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852

- Non-zero $b_k(T)$ for $k \geq 2$ signal deviation from ideal HRG
- EV interactions between baryons ($b \approx 1$ fm^3) reproduce lattice trend

Using estimators for radius of convergence

a) Ratio estimator:

$$r_n = \left| \frac{(2n+2)(2n+1)\chi_{2n}^B}{\chi_{2n+2}^B} \right|^{1/2}$$

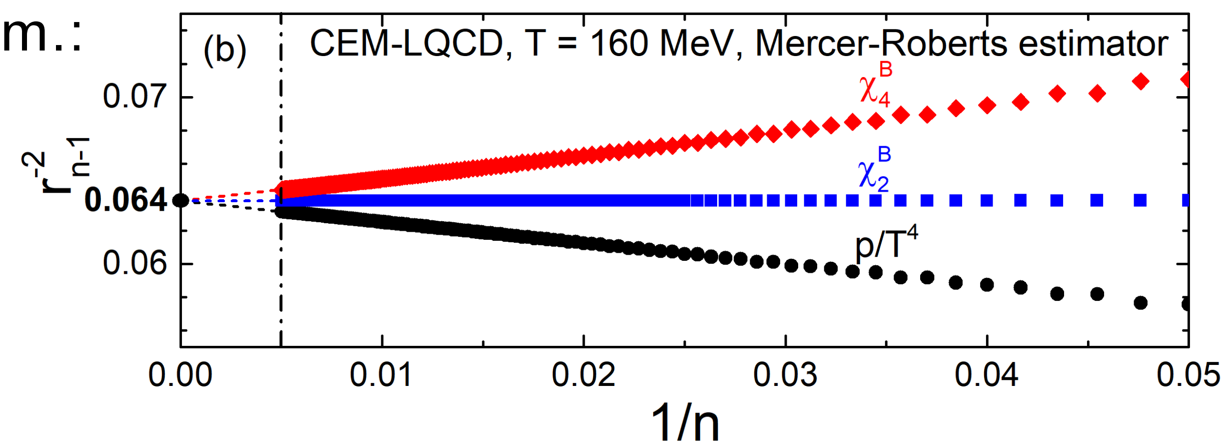


Ratio estimator is *unable* to determine the radius of convergence, nor to provide an upper or lower bound

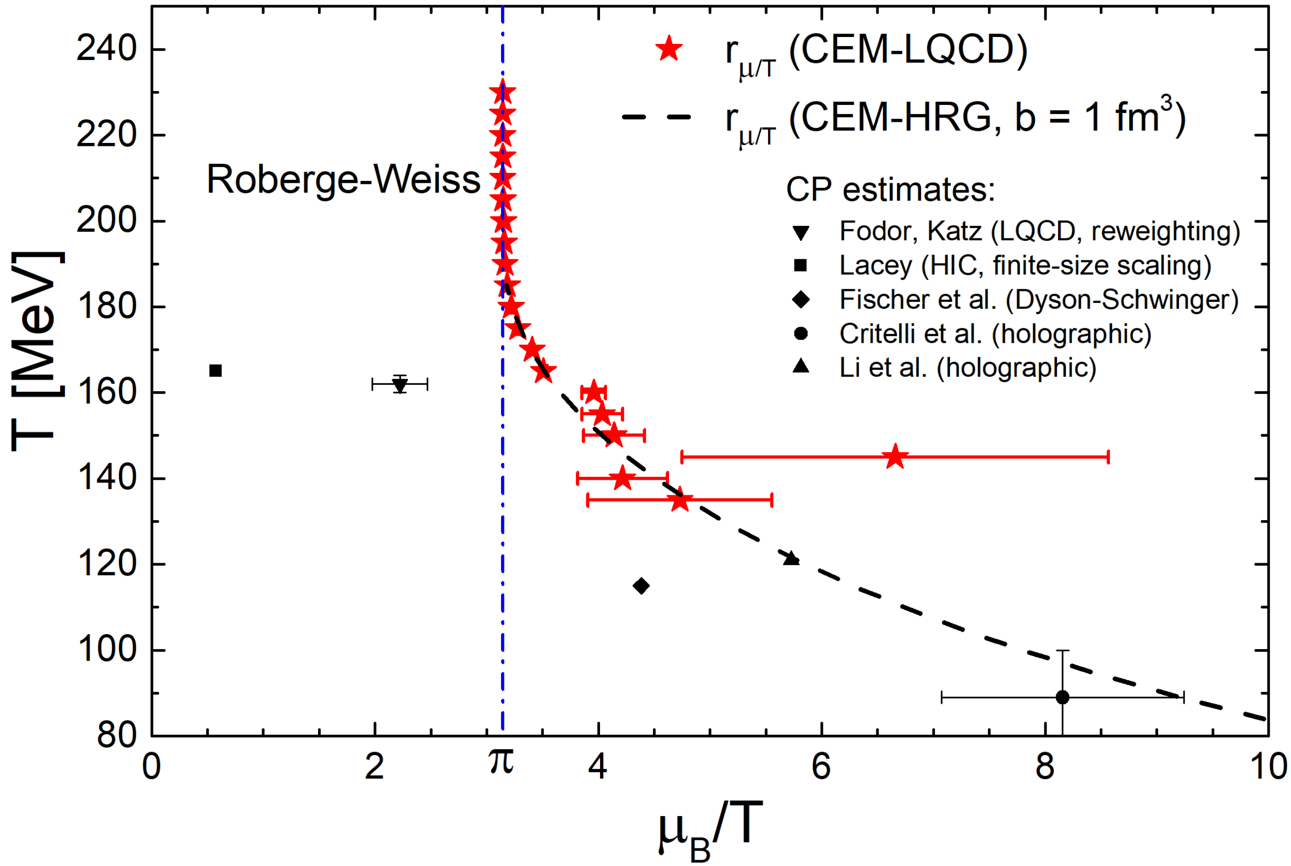
b) Mercer-Roberts estim.:

$$r_n = \left| \frac{c_{n+1}\, c_{n-1} - c_n^2}{c_{n+2}\, c_n - c_{n+1}^2} \right|^{1/4}$$

$$c_n = \frac{\chi_{2n}^B}{(2n)!}$$



CEM: Radius of convergence



Radius of convergence approaches Roberge-Weiss transition value

- At $T > T_{RW}$ expected $\left[\frac{\mu_B}{T}\right]_c = \pm i\pi$ [Roberge, Weiss, NPB '86] $T_{RW} \sim 208$ MeV [C. Bonati et al., 1602.01426]
- Complex plane singularities interfere with the search for CP

Expected asymptotics

- At low T/densities QCD $\simeq$ ideal hadron resonance gas

$$\frac{p^{\rm hrg}(T,\mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2\,\frac{\phi_B(T)}{T^3}\cosh\left(\frac{\mu_B}{T}\right),$$

$$\phi_B(T) = \sum_{i\in B}\int dm\,\rho_i(m)\,\frac{d_i\,m^2\,T}{2\pi^2}\,K_2\left(\frac{m}{T}\right),$$

$$p_0^{hrg}(T) = \frac{\phi_M(T)}{T^3},\quad p_1^{hrg}(T) = \frac{2\,\phi_B(T)}{T^3},\quad p_k^{\rm hrg}(T) \equiv 0,\ k\geq 2$$

- At high T QCD $\simeq$ ideal gas of massless quarks and gluons

$$\frac{p^{\rm SB}(T,\mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s}\left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_B}{3T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_B}{3T}\right)^4\right],$$

$$p_0^{\rm SB} = \frac{64\pi^2}{135},\quad p_k^{\rm SB} = \frac{(-1)^{k+1}}{k^2}\,\frac{4\,[3+4\,(\pi k)^2]}{27\,(\pi k)^2},\quad b_k^{SB} = k\,p_k^{\rm SB}.$$

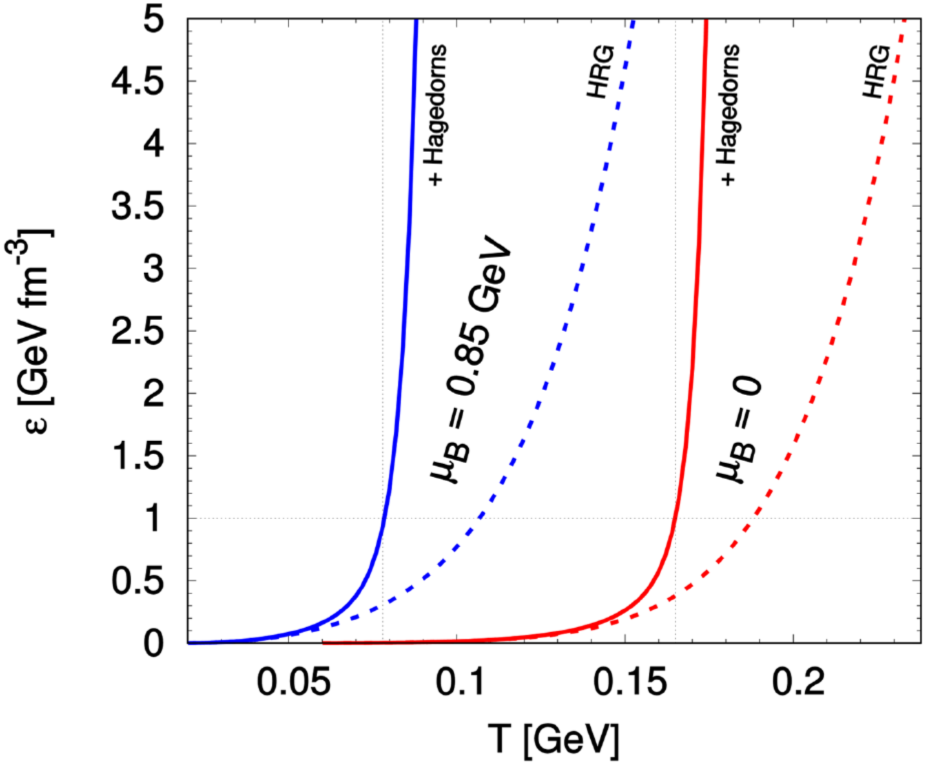
Lattice data explore intermediate, transition region $130 < T < 230$ *MeV*

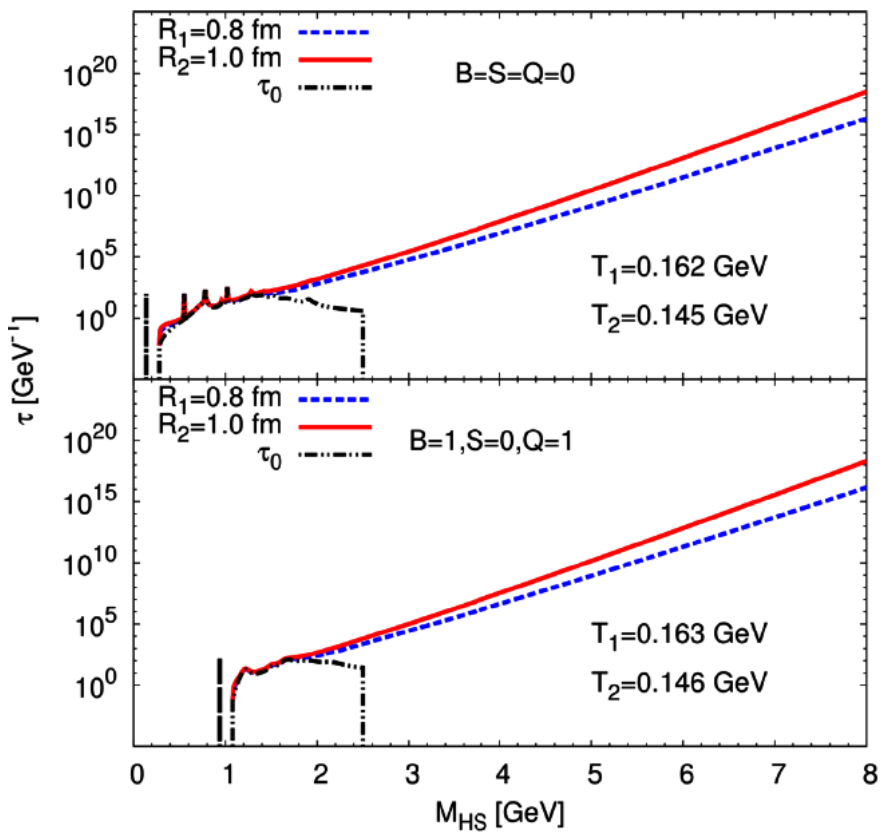
*In this study we assume that $\mu_S = \mu_Q = 0$

Hagedorn resonance gas

HRG + exponential Hagedorn mass spectrum, e.g. as obtained from the bootstrap equation [Hagedorn '65; Frautschi, '71]

$$\rho(m) = A\, m^{-\alpha} \exp(m/T_H)$$





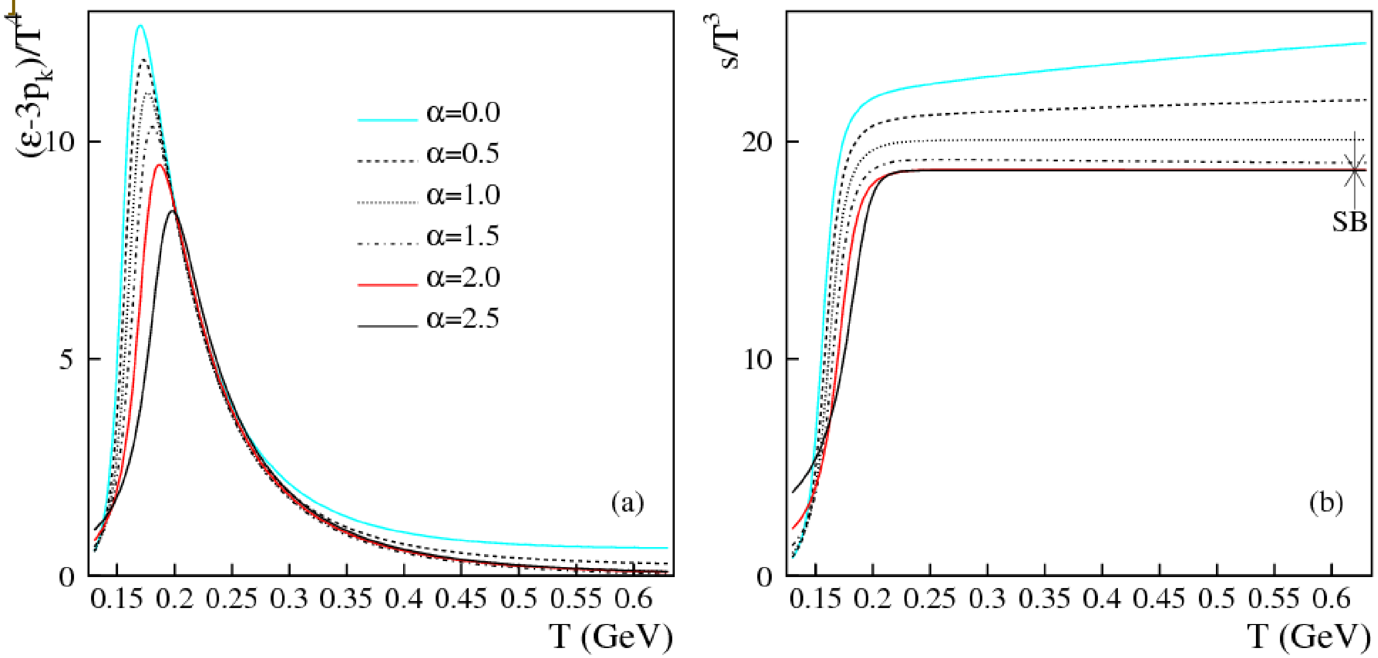
[Beitel, Gallmeister, Greiner, 1402.1458]

If Hagedorns are point-like, *T_H is the limiting temperature*

From limiting temperature to crossover

- A gas of extended objects → excluded volume
- Exponential spectrum of *compressible* QGP bags
- Both phases described by single partition function

[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; I. Zakout et al., NPA '07]



[Ferroni, Koch, PRC 79, 034905 (2009)]

Crossover transition in bag-like model qualitatively compatible with LQCD

Model formulation

Thermodynamic system of known hadrons and quark-gluon bags

Mass-volume density $\rho(m, v; \lambda_B, \lambda_Q, \lambda_S) = \rho_H + \rho_Q$

$$\rho_H(m, v; \lambda_B, \lambda_Q, \lambda_S) = \sum_{i \in \mathrm{HRG}} \lambda_B^{b_i} \lambda_Q^{q_i} \lambda_S^{s_i} d_i \, \delta(m - m_i) \, \delta(v - v_i)$$

PDG hadrons

$$\rho_Q(m, v; \lambda_B, \lambda_Q, \lambda_S) = C \, v^\gamma \, (m - Bv)^\delta \, \exp\left\{ \frac{4}{3} [\sigma_Q v]^{1/4} \, (m - Bv)^{3/4} \right\} \theta(v - V_0) \, \theta(m - Bv)$$

Quark-gluon bags [J. Kapusta, PRC '81; Gorenstein+, ZPC '84]

Non-overlapping particles $\rightarrow$ **isobaric (pressure) ensemble**

[Gorenstein, Petrov, Zinovjev, PLB '81]

$$\hat{Z}(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int_0^\infty Z(T, V, \lambda_B, \lambda_Q, \lambda_S) \, e^{-sV} \, dV = [s - f(T, s, \lambda_B, \lambda_Q, \lambda_S)]^{-1}$$

$$f(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int dv \int dm \, \rho(m, v; \lambda_B, \lambda_Q, \lambda_S) \, e^{-v s} \, \phi(T, m)$$

The system pressure is $p = Ts^*$ with s^* being the *rightmost* singularity of $\hat{Z}$

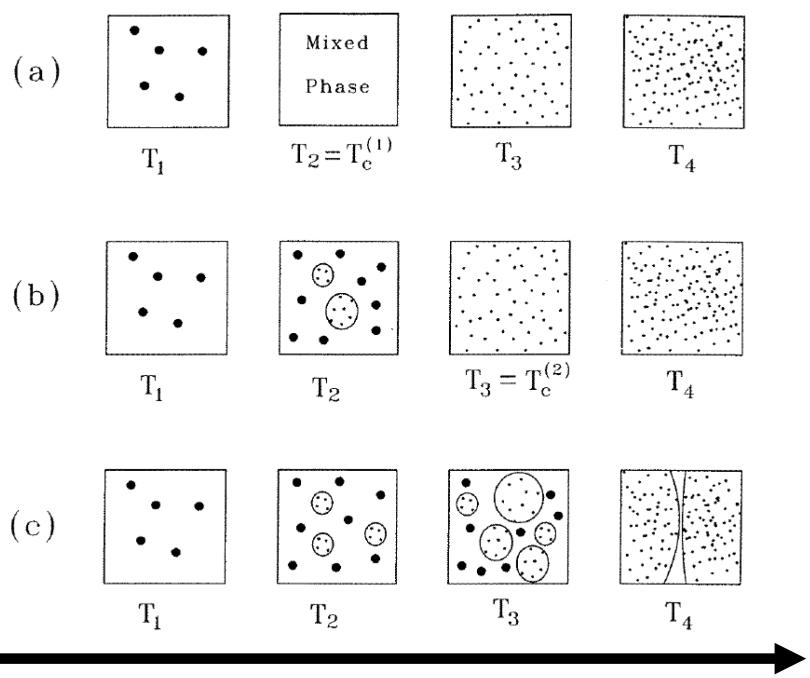
Mechanism for transition to QGP

The isobaric partition function, $\hat{Z}(T,s,\lambda) = [s - f(T,s,\lambda)]^{-1}$, has

- pole singularity $s_H = f(T, s_H, \lambda)$ **"hadronic" phase**
- singularity s_B in the function $f(T,s,\lambda)$ due to the exponential spectrum

$$p_B = T\, s_B = \frac{\sigma_Q}{3} T^4 - B$$

MIT bag model EoS for QGP [Chodos+, PRD '74; Baacke, APPB '77]



1st order PT

"collision" of singularities $s_H(T_C) = s_B(T_C)$

2nd order PT

crossover $s_H(T) > s_B(T)$ at all T

Crossover transition

Type of transition is determined by exponents γ and δ of bag spectrum

Crossover seen in lattice, realized in model for $\gamma + \delta \geq -3$ and $\delta \geq -7/4$ [Begun, Gorenstein, W. Greiner, JPG '09]

Transcendental equation for pr

$$p(T, \lambda_B, \lambda_Q, \lambda_S) = T \sum_{i \in \mathrm{HRG}} d_i \, \phi(T, m) \, \lambda_B^{b_i} \, \lambda_Q^{q_i} \, \lambda_S^{s_i} \, \exp\left(-\frac{m_i p}{4BT}\right)$$

$$+ \frac{C}{\pi} T^{5+4\delta} [\sigma_Q]^{\delta+1/2} [B + \sigma_Q T^4]^{3/2} \left(\frac{T}{p - p_B}\right)^{\gamma+\delta+3} \Gamma\left[\gamma + \delta + 3, \frac{(p - p_B) V_0}{T}\right]$$

Solved numerically

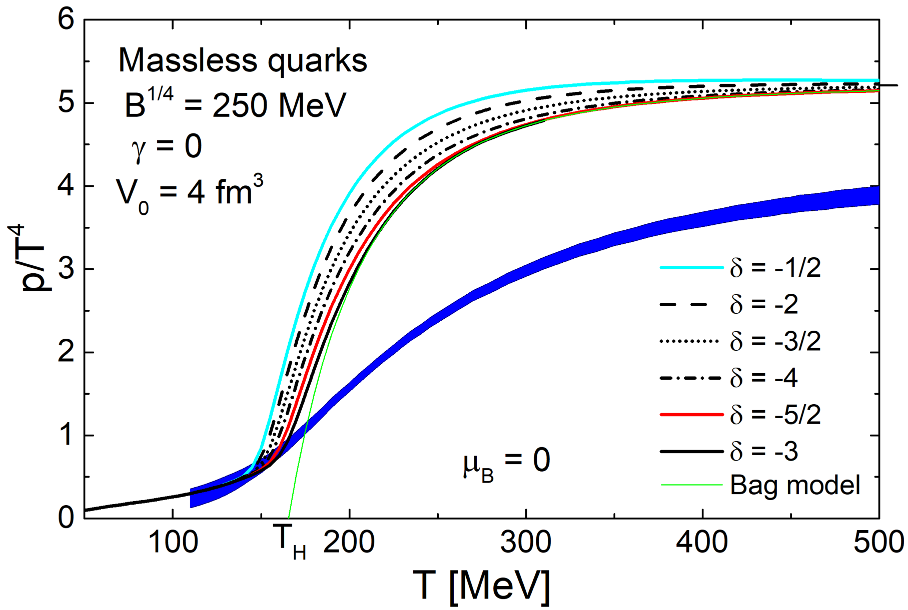
Calculation setup:

$$\gamma = 0, \quad -3 \leq \delta \leq -\frac{1}{2}, \quad B^{1/4} = 250 \text{ MeV}, \quad C = 0.03 \text{ GeV}^{-\delta+2}, \quad V_0 = 4 \text{ fm}^3$$

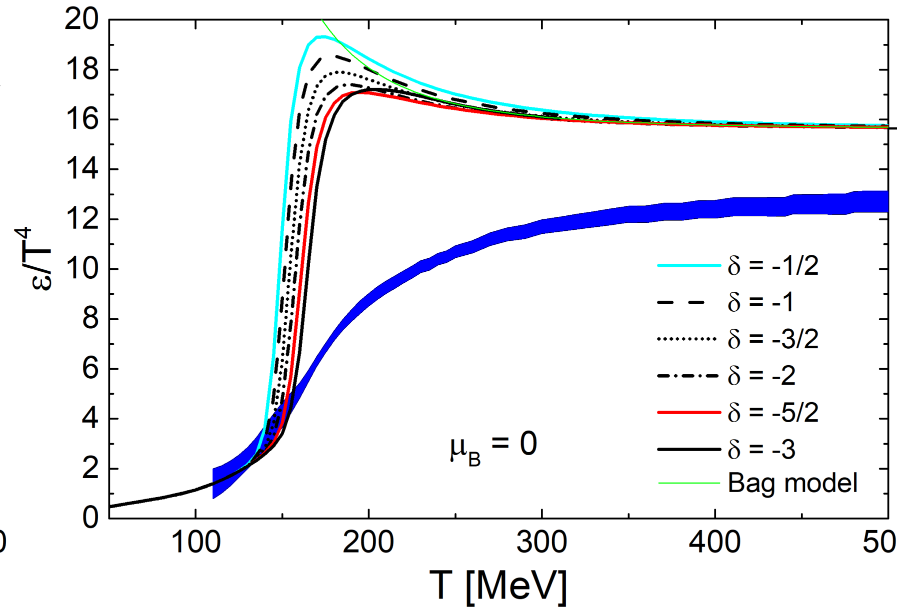
$$T_H = \left(\frac{3B}{\sigma_Q}\right)^{1/4} \simeq 165 \text{ MeV}$$

Thermodynamic functions

Pressure p/T^4



Energy density ε/T^4

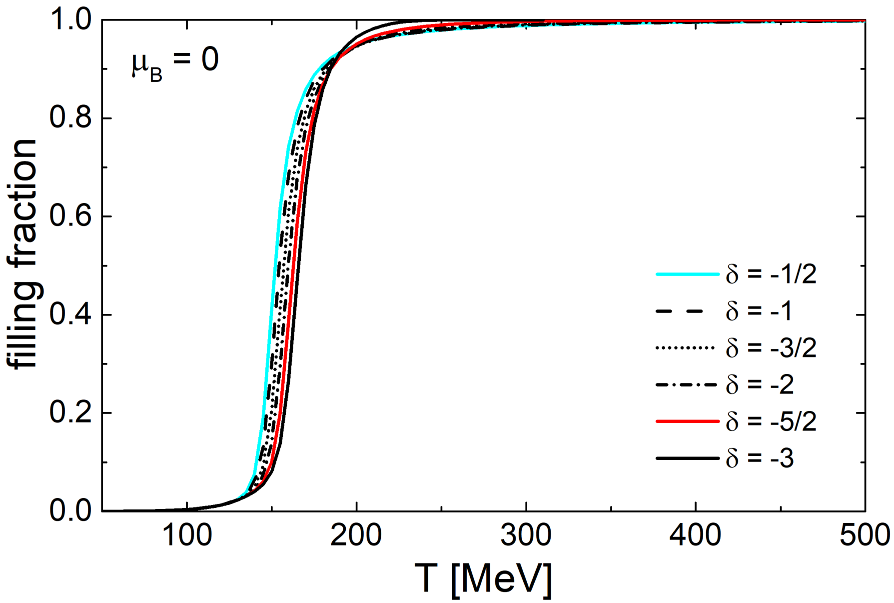


Lattice data from 1309.5258 (Wuppertal-Budapest)

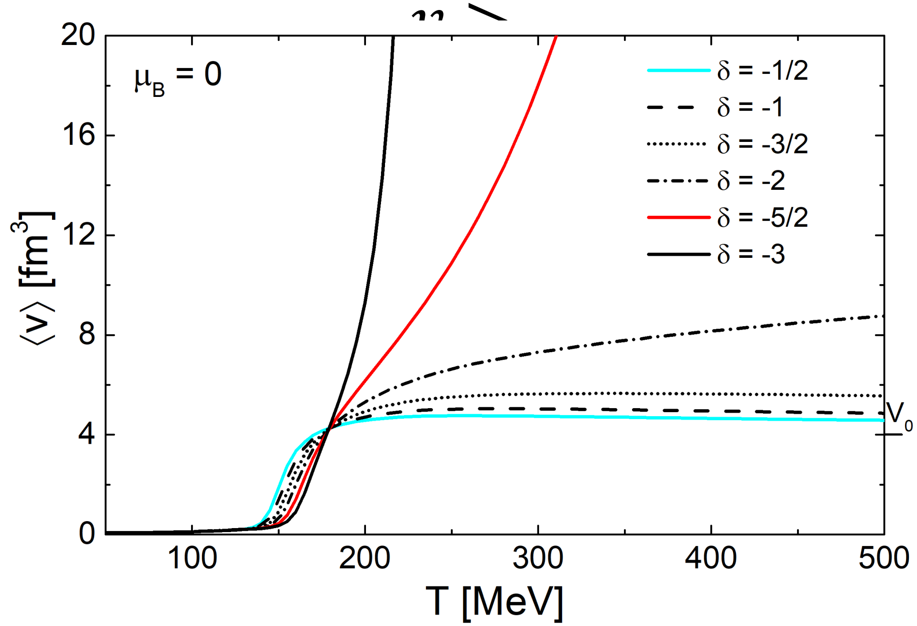
- Crossover transition towards bag model EoS
- Dependence on δ is mild
- Approach to the Stefan-Boltzmann limit is too fast
- Peak in energy density, not seen on the lattice

Nature of the transition

Filling fraction $= \frac{<V_{had}>}{V}$



Mean hadron volume <

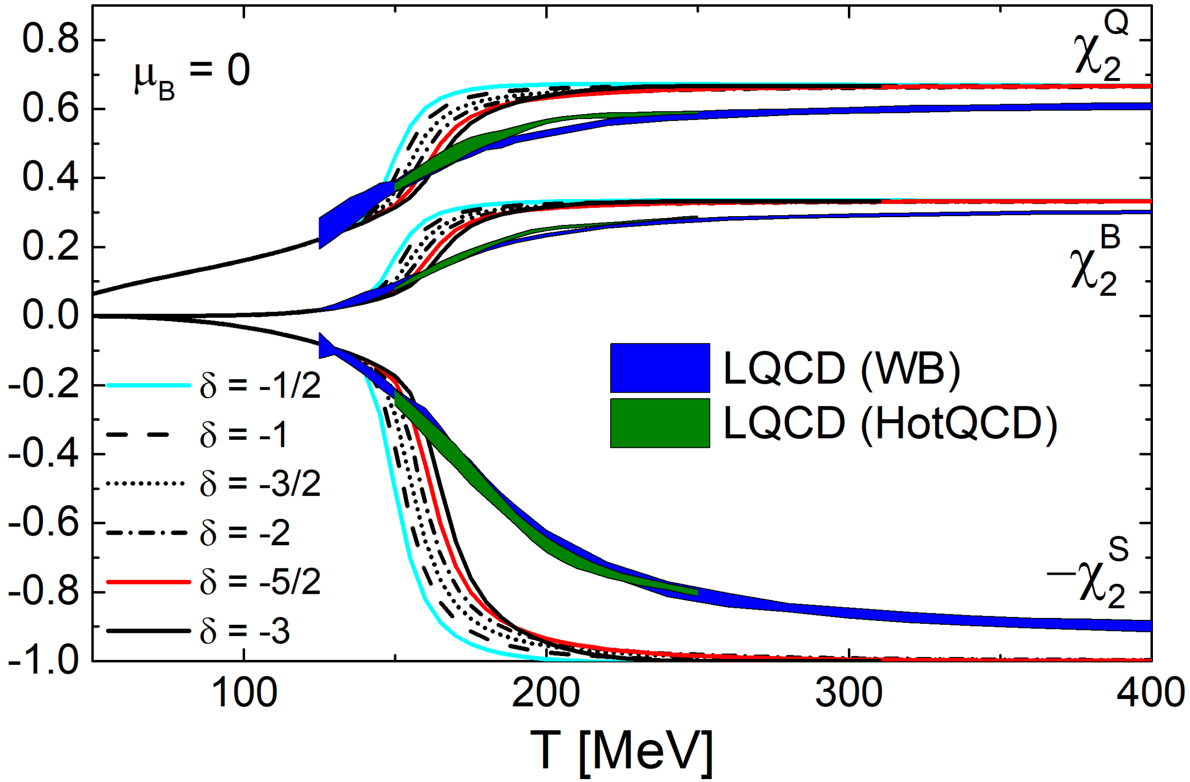


- Bags occupy almost whole space at large temperatures
- Strongest changes take place in the vicinity of T_H
- At $\delta < -7/4$ and $T \to \infty$ whole space — large bags with QGP

Conserved charges susceptibilities

$$\chi^{BSQ}_{lmn} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l \, \partial(\mu_S/T)^m \, \partial(\mu_Q/T)^n}$$

Available from lattice QCD, not considered in this type of model before



Qualitatively compatible with lattice QCD

Bag model with massive quarks

Main source of quantitative disagreement comes from inaccuracy of the standard MIT bag model with massless quarks for describing QGP

Quasiparticle models suggest sizable thermal masses of quarks and gluons in high-temperature QGP [Peshier et al., PLB '94; PRC '00; PRC '02]

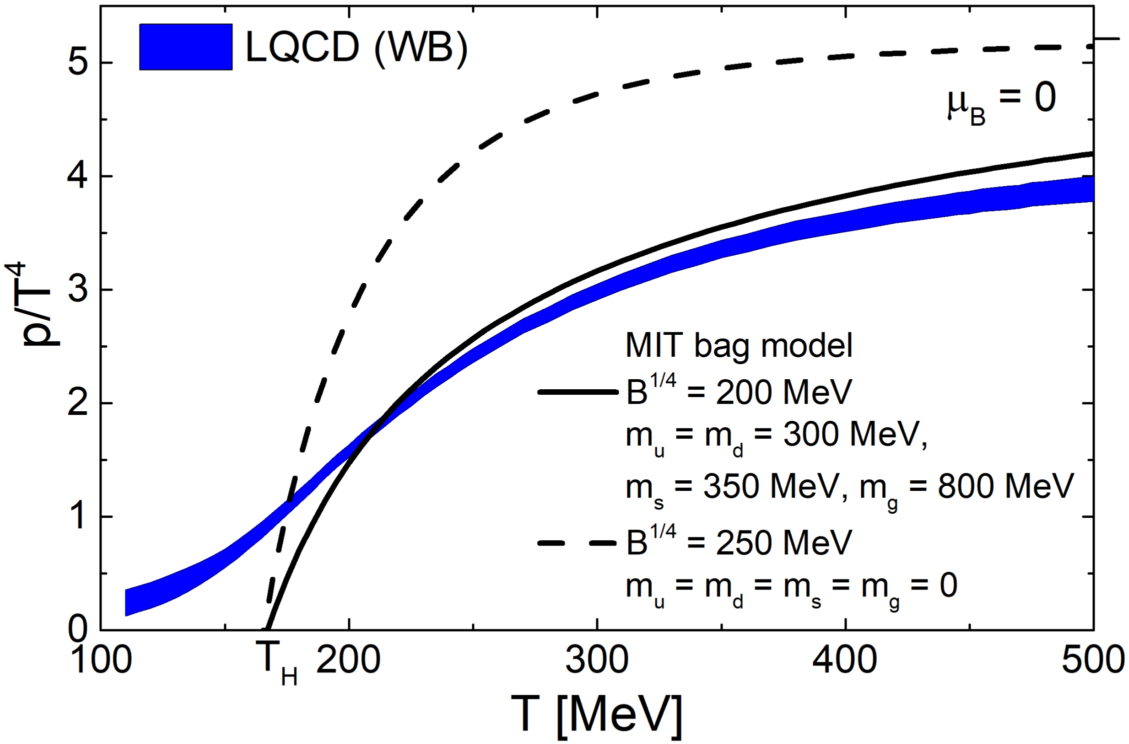
Heavy-bag model: bag model EoS with non-interacting *massive* quarks and gluons and the bag constant [Ivanov et al., PRC 72, 025804 (2005)]

Massive quarks and gluons instead of massless ones:

$$\sigma_Q(T,\lambda_B,\lambda_Q,\lambda_S) = \frac{8}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_g^2}}\left[\exp\left(\frac{\sqrt{k^2+m_g^2}}{T}\right)-1\right]^{-1}$$
$$+\sum_{f=u,d,s}\frac{3}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_f^2}}\left[\lambda_f^{-1}\exp\left(\frac{\sqrt{k^2+m_f^2}}{T}\right)+1\right]^{-1}$$
$$+\sum_{f=u,d,s}\frac{3}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_f^2}}\left[\lambda_f\exp\left(\frac{\sqrt{k^2+m_f^2}}{T}\right)+1\right]^{-1}$$

Bag model with massive quarks

Introduction of constituent masses leads to much better description of QGP



Parameters for the crossover model:

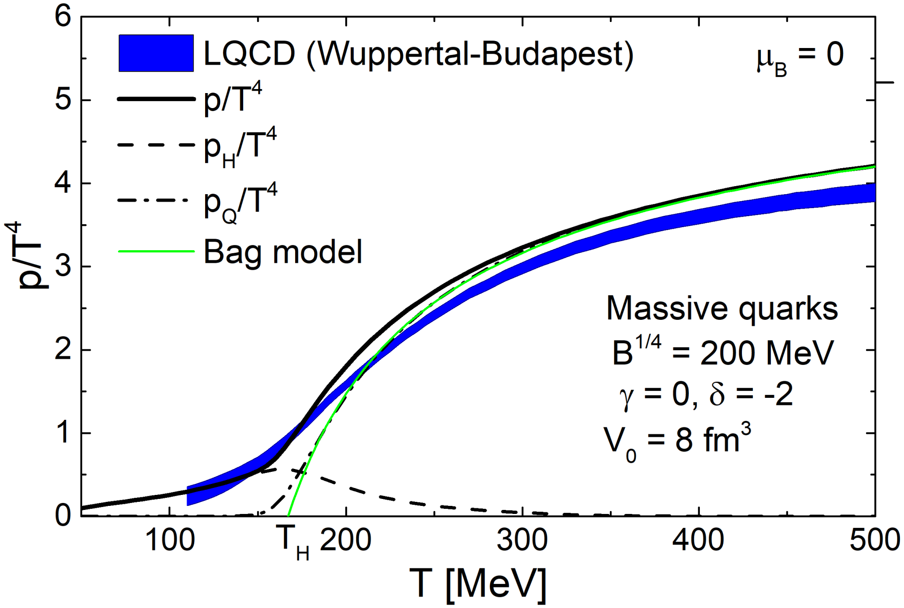
$$m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV}, \quad m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV}$$

$$\gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3$$

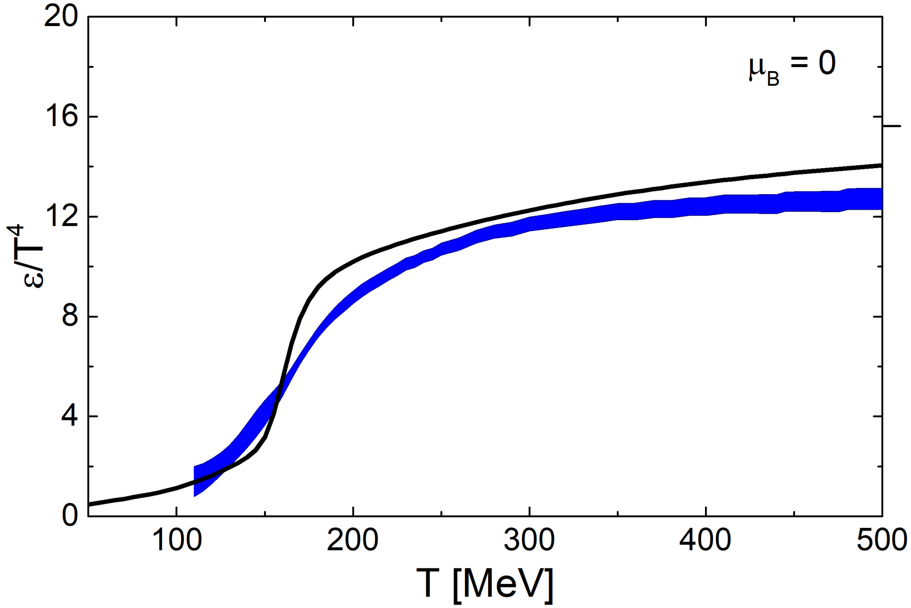
$$T_H \simeq 167 \text{ MeV}$$

Hagedorn model: Thermodynamic functions

Pressure p/T^4



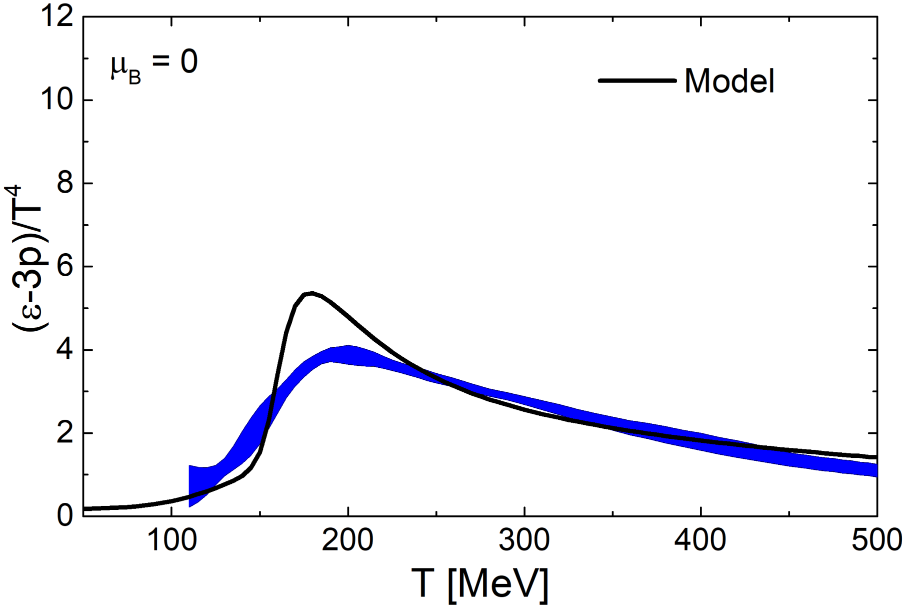
Energy density ε/T^4



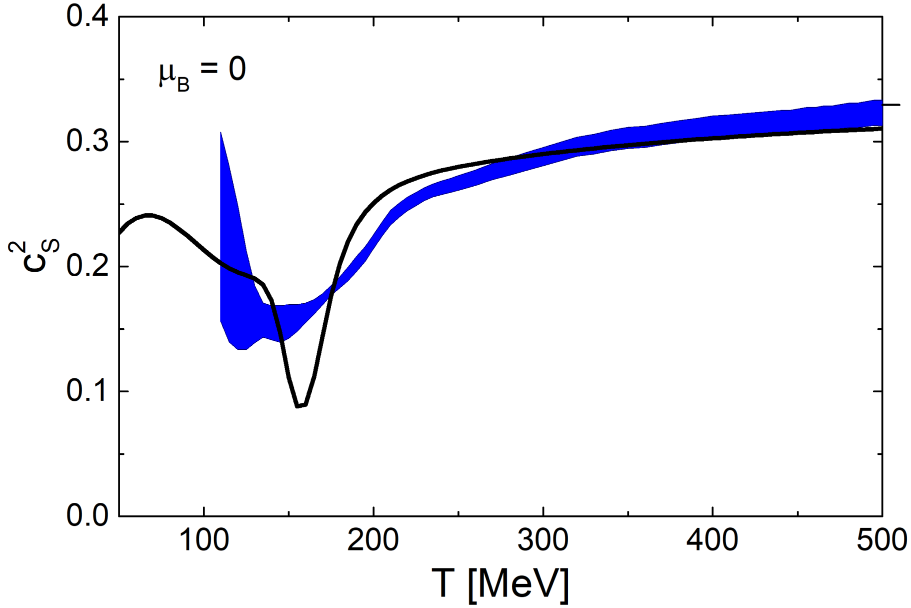
- Semi-quantitative description of lattice data
- Peak in energy density gone!

Hagedorn model: Thermodynamic functions

Trace anomaly $(\varepsilon - 3p)/T^4$

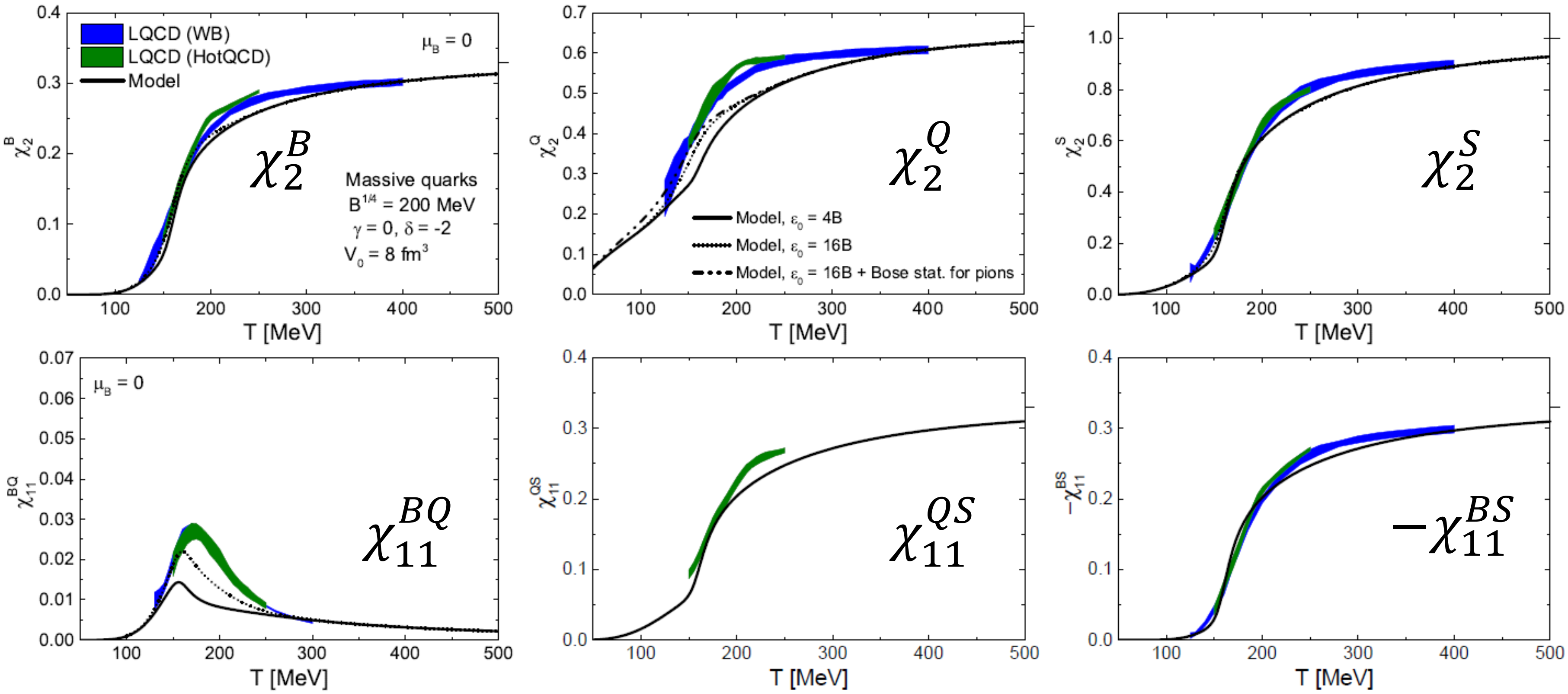


Speed of sound $c_s^2 = dp/d\varepsilon$



Hagedorn model: Susceptibilities

$$\chi^{BSQ}_{lmn} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l\,\partial(\mu_S/T)^m\,\partial(\mu_Q/T)^n}$$



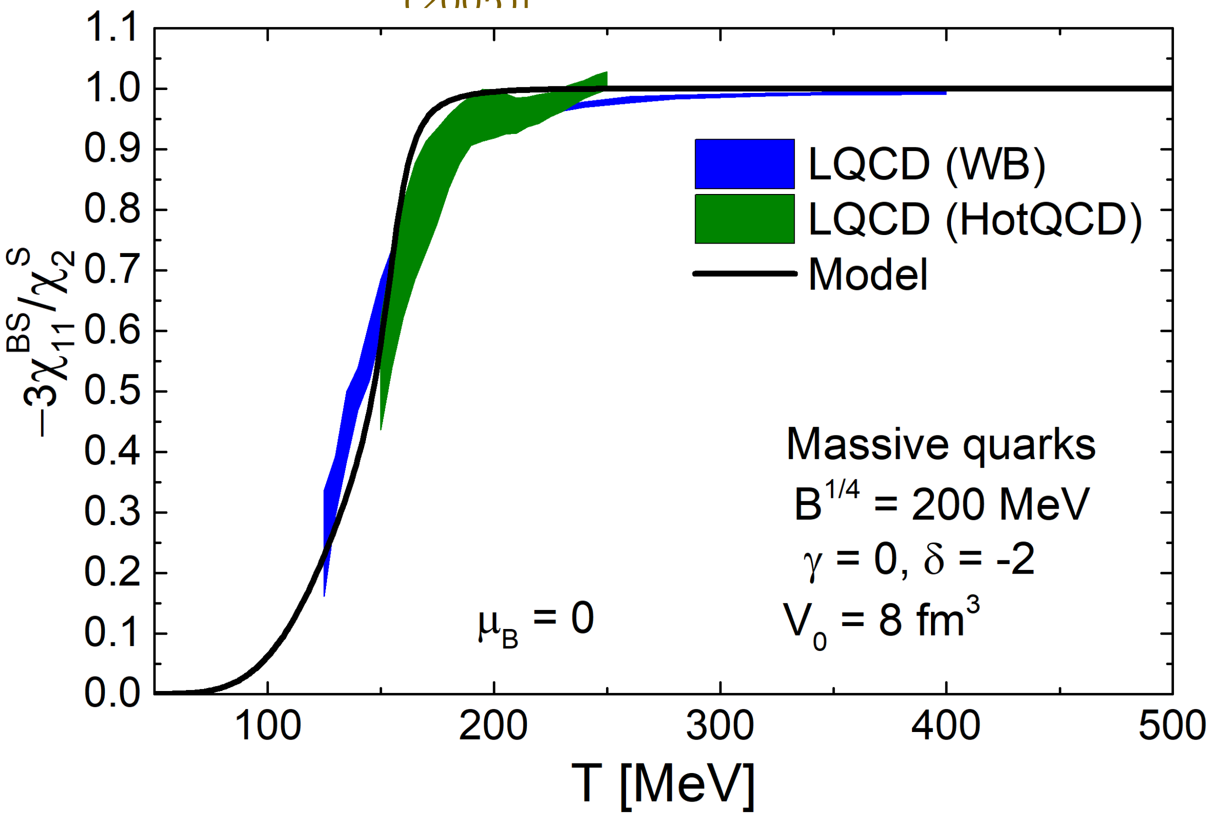
Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)

Hagedorn model: Baryon-strangeness ratio

$$C_{BS} = -\frac{3\chi_{11}^{BS}}{\chi_2^S}$$

Useful diagnostic of QCD matter

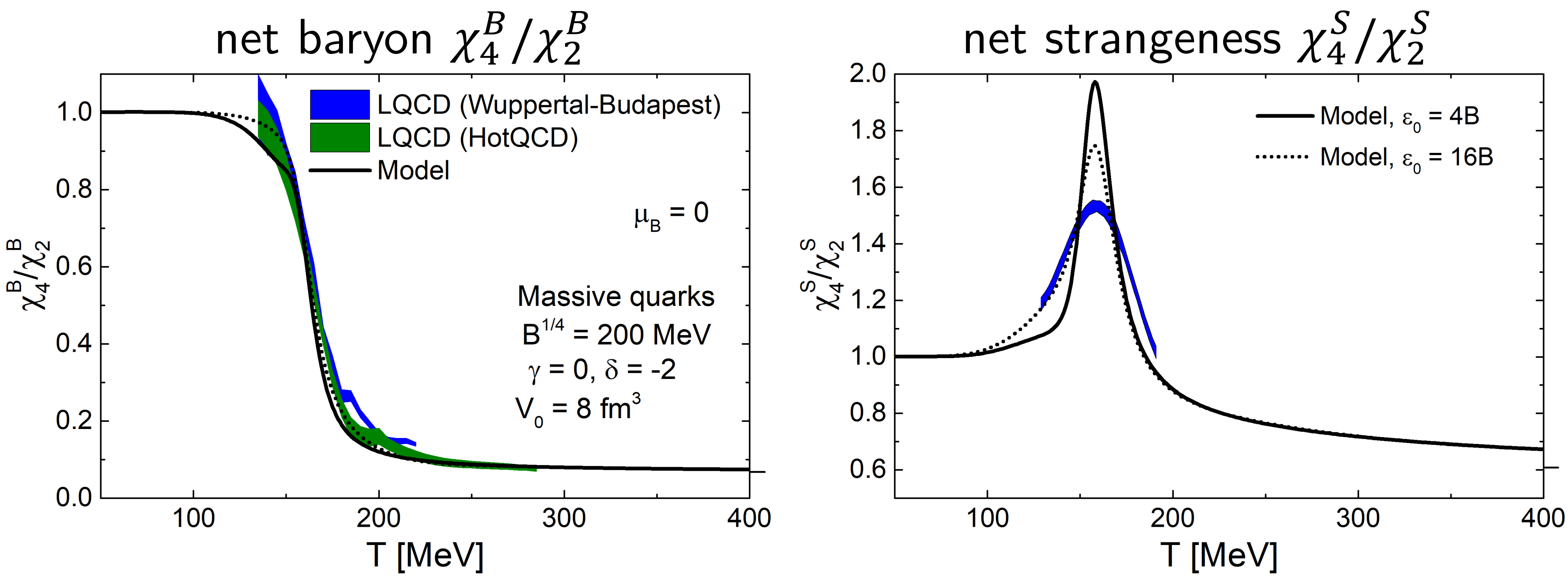
[V. Koch, Majumder, Randrup, PRL 95, 182301 (2005)]



Well consistent with lattice QCD

Hagedorn model: Higher-order susceptibilities

Higher-order susceptibilities are particularly sensitive probes of the parton-hadron transition and possible remnants of criticality at $\mu_B = 0$



Lattice data from 1305.6297 & 1805.04445 (Wuppertal-Budapest), 1708.04897 (HotQCD)

- Drop of χ_4^B/χ_2^B caused by repulsive interactions which ensure crossover transition to QGP
- Peak in χ_4^S/χ_2^S is an interplay of the presence of multi-strange hyperons and repulsive interactions