Phase transition at finite density and the cluster expansion in fugacities

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- Exactly solvable model with a phase transition
- Extracting information from Fourier coefficients

$$\sum_{k=1}^{\infty} b_k(T) \sinh\left(\frac{k\,\mu_B}{T}\right)$$

EMMI Workshop "Probing the Phase Structure of Strongly Interacting Matter: Theory and Experiment"

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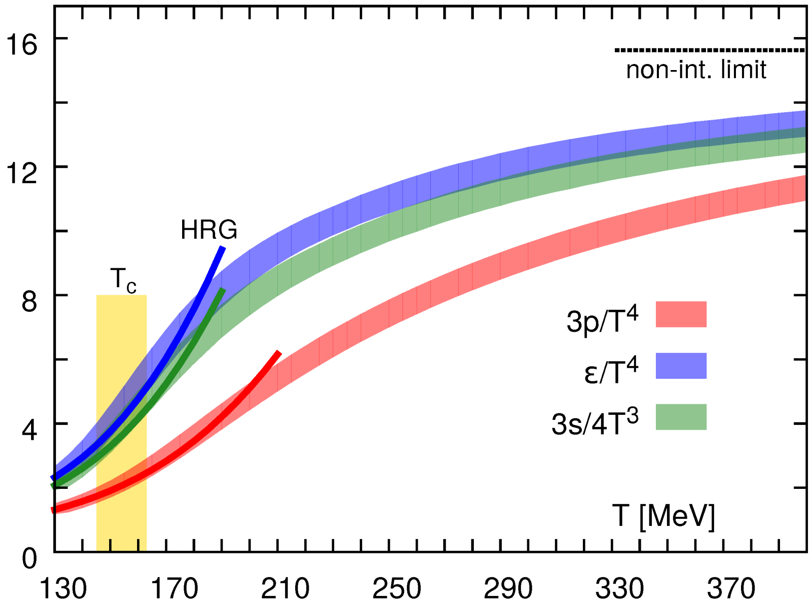
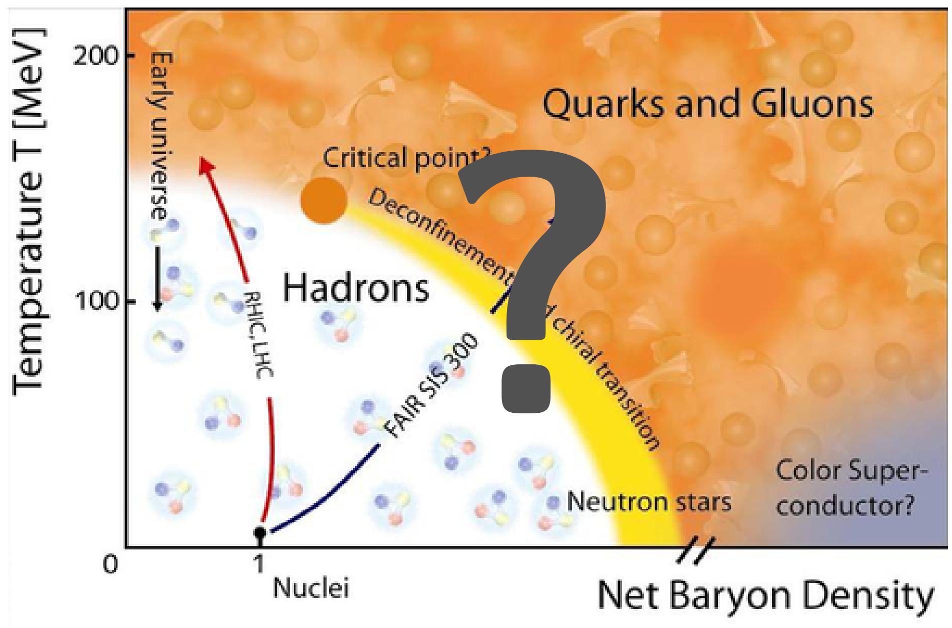
QCD phase diagram: towards finite density

?

$\mu_B = 0$ → $T - \mu_B$ plane





- QCD EoS at $\mu_B = 0$ available from lattice QCD
- Determination of phase structure at finite μ_B, in particular the critical point, is one of the major goals in the field

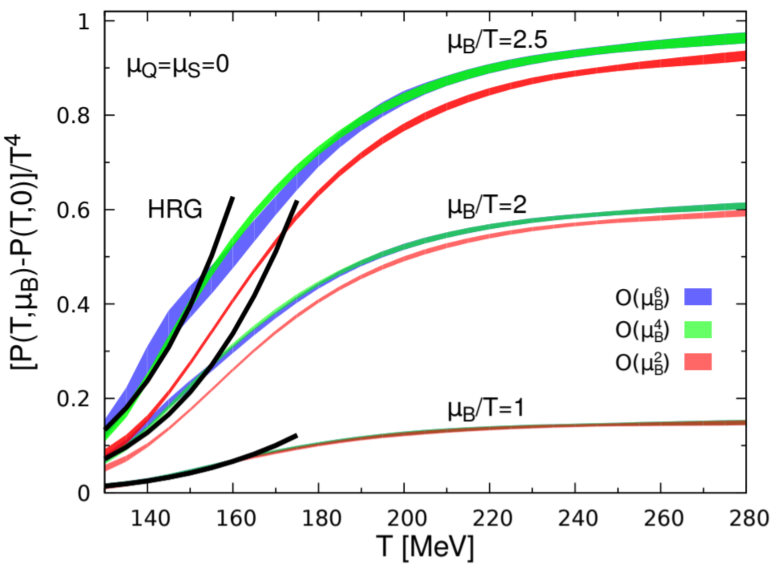
Common lattice-based methods for finite μ_B

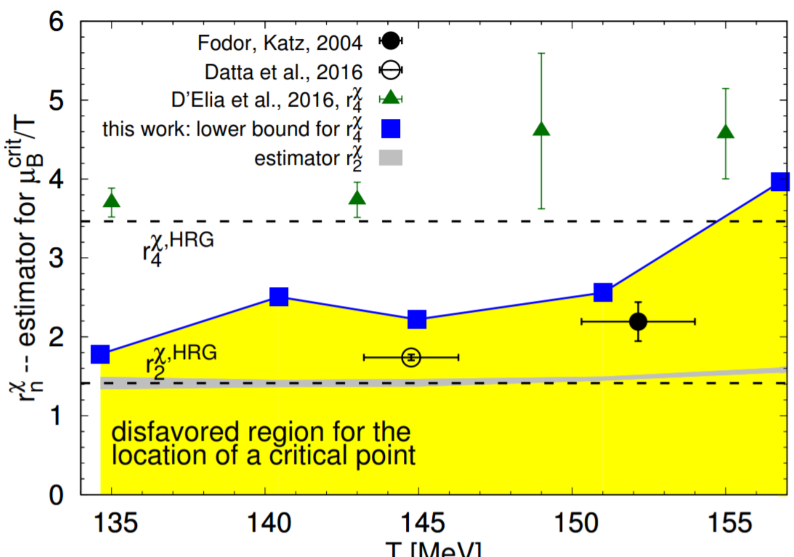
- Taylor expansion

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \frac{\chi_2^B(T,0)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T,0)}{4!}(\mu_B/T)^4 + \ldots$$

χ_k^B – cumulants of net baryon distribution, computed up to χ_8^B

[Wuppertal-Budapest collaboration, 1805.04445; etc.]





[HotQCD collaboration, 1701.04325]

No hints for a CP from χ_k^B, "small" $\mu_B/T < 2$ disfavored

- Other methods: **analytic continuation (Im $\boldsymbol{\mu_B}$)**, reweighing, etc.

Cluster expansion in fugacities

Expand in fugacity $\lambda_B = e^{\mu_B/T}$ instead of μ_B/T – a relativistic analogue of Mayer's cluster expansion:

$$\frac{p(T,\mu_B)}{T^4} = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_{|k|}(T)\, e^{k\mu_B/T} = \frac{p_0(T)}{2} + \sum_{k=1}^{\infty} p_k(T)\cosh(k\mu_B/T)$$

Net baryon density: $\dfrac{\rho_B(T,\mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T)\sinh(k\mu_B/T), \quad b_k \equiv k p_k$

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Analytic continuation to imaginary μ_B yields trigonometric Fourier series

$$\frac{\rho_B(T,\mathrm{i}\tilde{\mu}_B)}{T^3} = \mathrm{i}\sum_{k=1}^{\infty} b_k(T)\,\sin\left(\frac{k\,\tilde{\mu}_B}{T}\right)$$

with Fourier coefficients $$b_k(T) = \frac{2}{\pi\, T^4}\int_0^{\pi T} d\tilde{\mu}_B\,[\mathrm{Im}\,\rho_B(T,i\tilde{\mu}_B)]\,\sin(k\,\tilde{\mu}_B/T)$$

Four leading coefficients b_k computed in LQCD at the physical point

[**V.V.**, A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852]

Why cluster expansion is interesting?

Convergence properties of cluster expansion determined by singularities of thermodynamic potential in complex fugacity plane $\rightarrow$ encoded in the asymptotic behavior of the Fourier coefficients b_k

Examples:

- ideal quantum gas $\quad b_k \sim (\pm 1)^{k-1} \dfrac{e^{-km/T}}{k^{3/2}}$ *Bose-Einstein condensation*
- cluster expansion model $\quad b_k \sim (-1)^{k-1} \dfrac{|\lambda_{\rm br}|^{-k}}{k}$ $|\lambda_{br}| = 1 \rightarrow$ *Roberge-Weiss transition at imaginary* μ_B
  [**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]
- excluded volume model $\quad b_k \sim (-1)^{k-1} \dfrac{|\lambda_{\rm br}|^{-k}}{k^{1/2}}$ *No phase transition, but a singularity at a negative* λ
  [Taradiy, **V.V.**, Gorenstein, Stoecker, in preparation]
- chiral crossover $\quad b_k \sim \dfrac{e^{-k\tilde{\mu}_c}}{k^{2-\alpha}} \sin(k\theta_c + \theta_0)$ *Remnants of chiral criticality at* $\mu_B = 0$
  [Almasi, Friman, Morita, Redlich, 1902.05457]

This work: signatures of a CP and a phase transition at finite density

A model with a phase transition

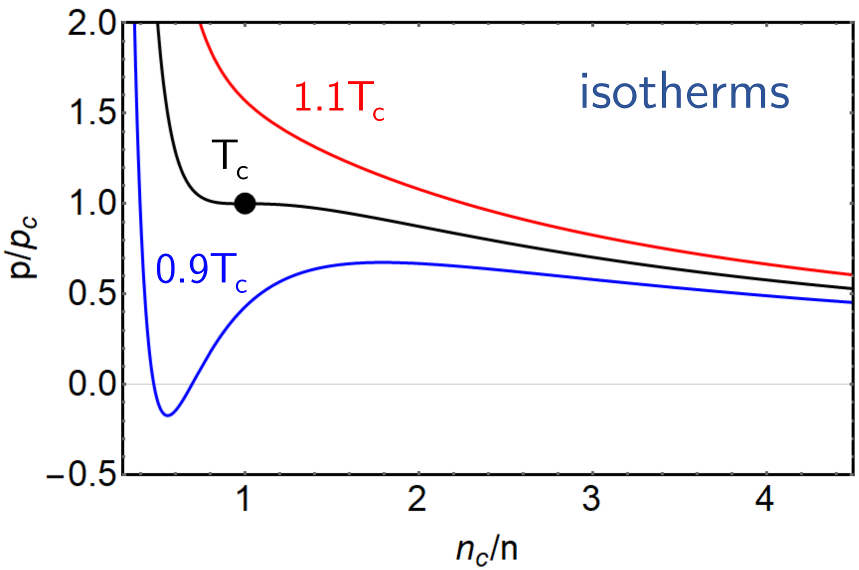
Our starting point is a single-component fluid. We are looking for a theory with a phase transition where Mayer's cluster expansion

$$\frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

can be worked out explicitly. The **"tri-virial" model (TVM)**

$$\boxed{p(T,n) = T\,n + T\left(b - \frac{a}{T}\right) n^2 + T\,b^2\,n^3}$$

which is the vdW equation truncated at n^3, has the required features.



Critical point:

$$\left(\frac{\partial p}{\partial n}\right)_T = 0, \qquad \left(\frac{\partial^2 p}{\partial n^2}\right)_T = 0.$$

$$T_c = \frac{\sqrt{3}-1}{2}\,\frac{a}{b}, \qquad n_c = \frac{1}{\sqrt{3}\,b}, \qquad p_c = \frac{3-\sqrt{3}}{18}\,\frac{a}{b^2}.$$

TVM in the grand canonical ensemble (GCE)

Transformation from (T, n) variables to (T, μ) [or (T, λ)] variables

$$p(T, n) = T\, n + T\, \left(b - \frac{a}{T}\right) n^2 + T\, b^2\, n^3$$

$$\Downarrow$$

$$p(T, n) = -\left(\frac{\partial F}{\partial V}\right)_{T,N} \quad \Rightarrow \quad F(T, V, N) \quad \Rightarrow \quad \mu = \left(\frac{\partial F}{\partial N}\right)_{T,V}$$

$$\Downarrow$$

$$\boxed{\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right], \qquad \lambda \equiv e^{\mu/T}}$$

The defining transcendental equation for the GCE particle number density $n(T, \lambda)$

This equation encodes the analytic properties of the grand potential associated with a phase transition

TVM: the branch points

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right]$$

The defining equation permits multiple solutions therefore $n(T, \lambda)$ is multi-valued and has singularities – the **branch points:**

$$\left(\frac{\partial \lambda}{\partial n}\right)_T = 0 \qquad \Rightarrow \qquad 3(bn_{\mathrm{br}})^2 + 2\left(1 - \frac{a}{bT}\right) bn_{\mathrm{br}} + 1 = 0$$

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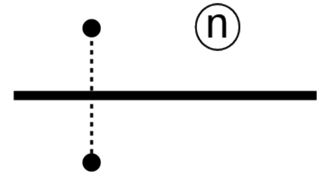
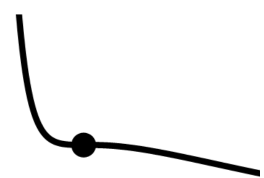
Solutions:

- $T > T_C$: two c.c. roots $n_{\mathrm{br1}} = (n_{\mathrm{br2}})^*$



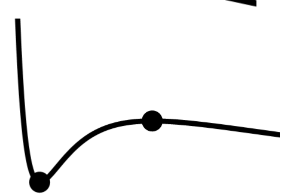
crossover singularities

- $T = T_C$: $n_{\mathrm{br1}} = n_{\mathrm{br2}} = n_{\mathrm{c}}$



the critical point

- $T < T_C$: two real roots n_{sp1} and n_{sp2}



the spinodal points

see also M. Stephanov, Phys. Rev. D 73, 094508 (2006)

TVM: Mayer's cluster expansion

$$\lambda = \frac{n}{\phi(T)} \exp\left[\frac{3}{2}(bn)^2 + 2n\left(b - \frac{a}{T}\right)\right] \xrightarrow{?} \frac{n(T,\lambda)}{T^3} = \frac{1}{2}\sum_{k=1}^{\infty} b_k(T)\,\lambda^k$$

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Lagrange inversion theorem

If $y=f(x)$, $y_0=f(x_0)$, $f'(x_0)\neq 0$, then

3.6.6

$$x=x_0+\sum_{k=1}^{\infty}\frac{(y-y_0)^k}{k!}\left[\frac{d^{k-1}}{dx^{k-1}}\left\{\frac{x-x_0}{f(x)-y_0}\right\}^k\right]_{x=x_0}$$

from Abramowitz, Stegun, "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables"

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$$y \equiv \lambda, \quad x \equiv n, \quad f(x) \equiv \lambda(n;T)$$

$$\lambda_0 = 0, \quad n_0 = 0$$

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$$y \equiv \lambda, \quad x \equiv n, \quad f(x) \equiv \lambda(n;T)$$

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Result:

$$b_k(T) = 2\,\frac{\phi(T)}{T^3}\,[b\,\phi(T)]^{k-1}\,\frac{1}{k!}\left(\frac{3k}{2}\right)^{\frac{k-1}{2}} \lim_{x\to 0}\frac{d^{k-1}}{dx^{k-1}} \exp\left[-2\left(1-\frac{a}{bT}\right)\sqrt{\frac{2k}{3}}\,x - x^2\right]$$

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Generating function of Hermite polynomials: $$e^{2tx-\frac{1}{2}x^2} = \sum_{n=0}^{\infty} H_n(t)\,\frac{x^n}{n!}$$

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The potentially non-trivial behavior of cluster integrals b_k associated with a presence of a phase transition is determined by the Hermite polynomials

Asymptotic behavior of cluster integrals

Asymptotic behavior of b_k determined mainly by Hermite polynomials

$$b_k \sim H_{k-1}\left[-\sqrt{\frac{2k}{3}}\left(1-\frac{a}{b\,T}\right)\right]$$

A catch: both the argument and the index of H tend to large values.

Asymptotic behavior of cluster integrals

Asymptotic behavior of b_k determined mainly by Hermite polynomials

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A catch: both the argument and the index of H tend to large values. Such a case was analyzed in **[D. Dominici, arXiv:math/0601078]**

1) $x > \sqrt{2n}$

$T < T_c$

$$H_n(x) \overset{n\to\infty}{\simeq} \exp\left[\frac{x^2-\sigma x-n}{2}+n\ln(\sigma+x)\right]\sqrt{\frac{1}{2}\left(1+\frac{x}{\sigma}\right)}, \qquad \sigma=\sqrt{x^2-2n}$$

2) $x \approx \sqrt{2n}$

$T = T_c$

$$H_n(x) \overset{n\to\infty}{\simeq} \exp\left[\frac{n}{2}\ln(2n)-\frac{3}{2}n+\sqrt{2n}\,x\right]\sqrt{2\pi}\,n^{1/6}\,\mathrm{Ai}\left[\sqrt{2}\,(x-\sqrt{2n})\,n^{1/6}\right]$$

3) $|x| < \sqrt{2n}$

$T > T_c$

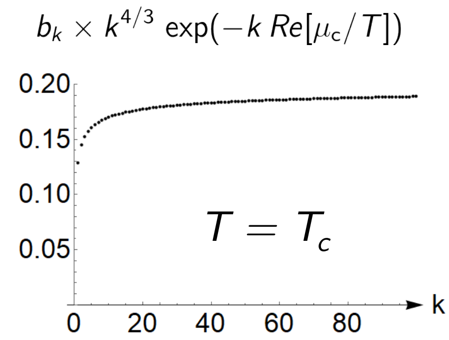
$$H_n\left[\sqrt{2n}\sin\theta\right] \overset{n\to\infty}{\simeq} \sqrt{\frac{2}{\cos\theta}}\exp\left\{\frac{n}{2}\left[\ln(2n)-\cos(2\theta)\right]\right\}\cos\left\{n\left[\frac{1}{2}\sin(2\theta)+\theta-\frac{\pi}{2}\right]+\frac{\theta}{2}\right\}$$

Asymptotic behavior changes as one traverses the critical temperature

Asymptotic behavior of cluster integrals

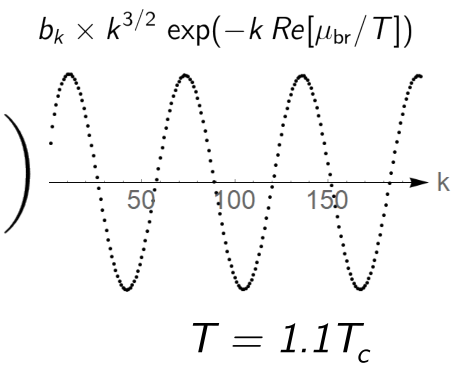
1) $T < T_c$: $$b_k(T) \overset{k\to\infty}{\simeq} A_- \frac{e^{-\frac{k\,\mu_{\mathrm{sp1}}}{T}}}{k^{3/2}}$$

b_k see the spinodal point of a first-order phase transition



2) $T = T_c$: $$b_k(T) \overset{k\to\infty}{\simeq} A_c \frac{e^{-\frac{k\,\mu_c}{T}}}{k^{4/3}}$$

b_k see the critical point



3) $T > T_c$: $$b_k(T) \overset{k\to\infty}{\simeq} A_+ \frac{e^{-\frac{k\,\mu^R_{\mathrm{br}}}{T}}}{k^{3/2}} \sin\left(k\,\frac{\mu^I_{\mathrm{br}}}{T} + \frac{\theta_0}{2}\right)$$

crossover singularities $\rightarrow$ oscillatory behavior of b_k

Behavior expected to be universal for the mean-field universality class, the likely effect of a change in universality class (e.g. 3D-Ising) is a modification of the power-law exponents

Applications to the QCD thermodynamics

TVM for "baryonic" pressure: $p_B(T,n) = T\,n_B + T\left(b - \frac{a}{T}\right) n_B^2 + T\,b^2\,n_B^3$

Symmetrization: $\mu_B \to -\mu_B$

$$p = \underbrace{p_B(T,\mu_B)}_{\text{"baryons"}} + \underbrace{p_B(T,-\mu_B)}_{\text{"anti-baryons"}} + \underbrace{p_M(T)}_{\text{"mesons"}}$$

$$\frac{\rho_B(T,\mathrm{i}\tilde{\mu}_B)}{T^3} = \mathrm{i}\sum_{k=1}^{\infty} b_k(T)\,\sin\left(\frac{k\,\tilde{\mu}_B}{T}\right)$$

Cluster integrals become Fourier coefficients *(as long as $b_k(T) \overset{k\to\infty}{\to} 0$ holds)*

Riemann-Lebesgue lemma

Expected asymptotics

$$b_k(T) \overset{k\to\infty}{\simeq} A\,\frac{e^{-\frac{k\,\mu_{\mathrm{br}}^R}{T}}}{k^\alpha}\,\sin\left(k\,\frac{\mu_{\mathrm{br}}^I}{T} + \frac{\theta_0}{2}\right), \quad \frac{\mu_{\mathrm{br}}^R}{T} = \mathrm{Re}\left[\frac{\mu_B}{T}\right]_{\mathrm{br}}, \quad \frac{\mu_{\mathrm{br}}^I}{T} = \mathrm{Im}\left[\frac{\mu_B}{T}\right]_{\mathrm{br}}$$

Can be tested in lattice QCD at imaginary chemical potential

Extracting information from Fourier coefficients

$$b_k(T) \overset{k\to\infty}{\simeq} A \frac{e^{-\frac{k\,\mu_{\rm br}^R}{T}}}{k^\alpha} \sin\left(k\,\frac{\mu_{\rm br}^I}{T} + \frac{\theta_0}{2}\right)$$

Real part of the limiting singularity determines the exponential suppression of Fourier coefficients

To extract $\mathrm{Re}[\mu_{\rm br}/T]$ fit b_k with $\log|b_k| = A - (3/2)\log k - k\,\mathrm{Re}\left[\frac{\mu_{\rm br}}{T}\right]$

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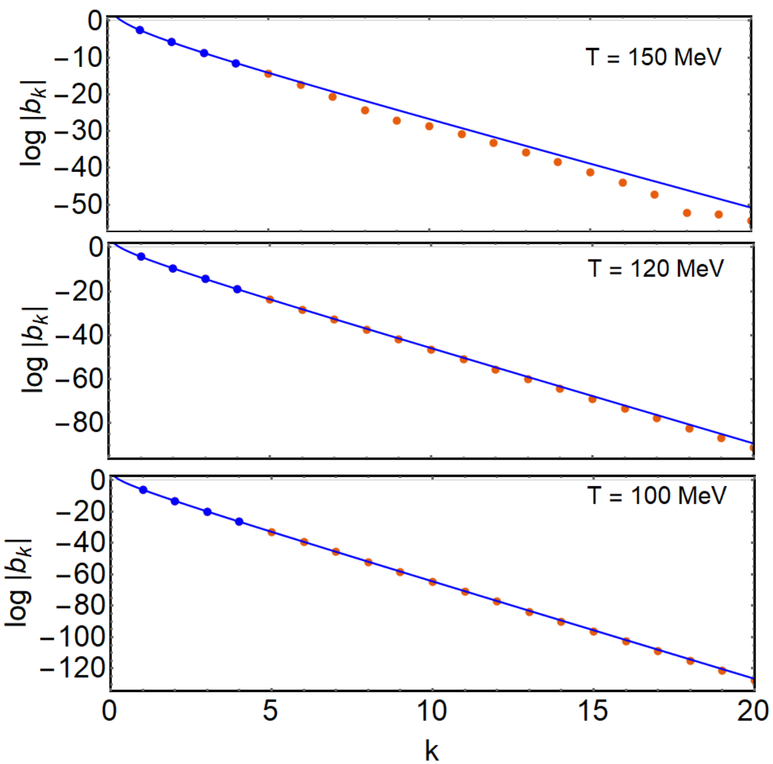
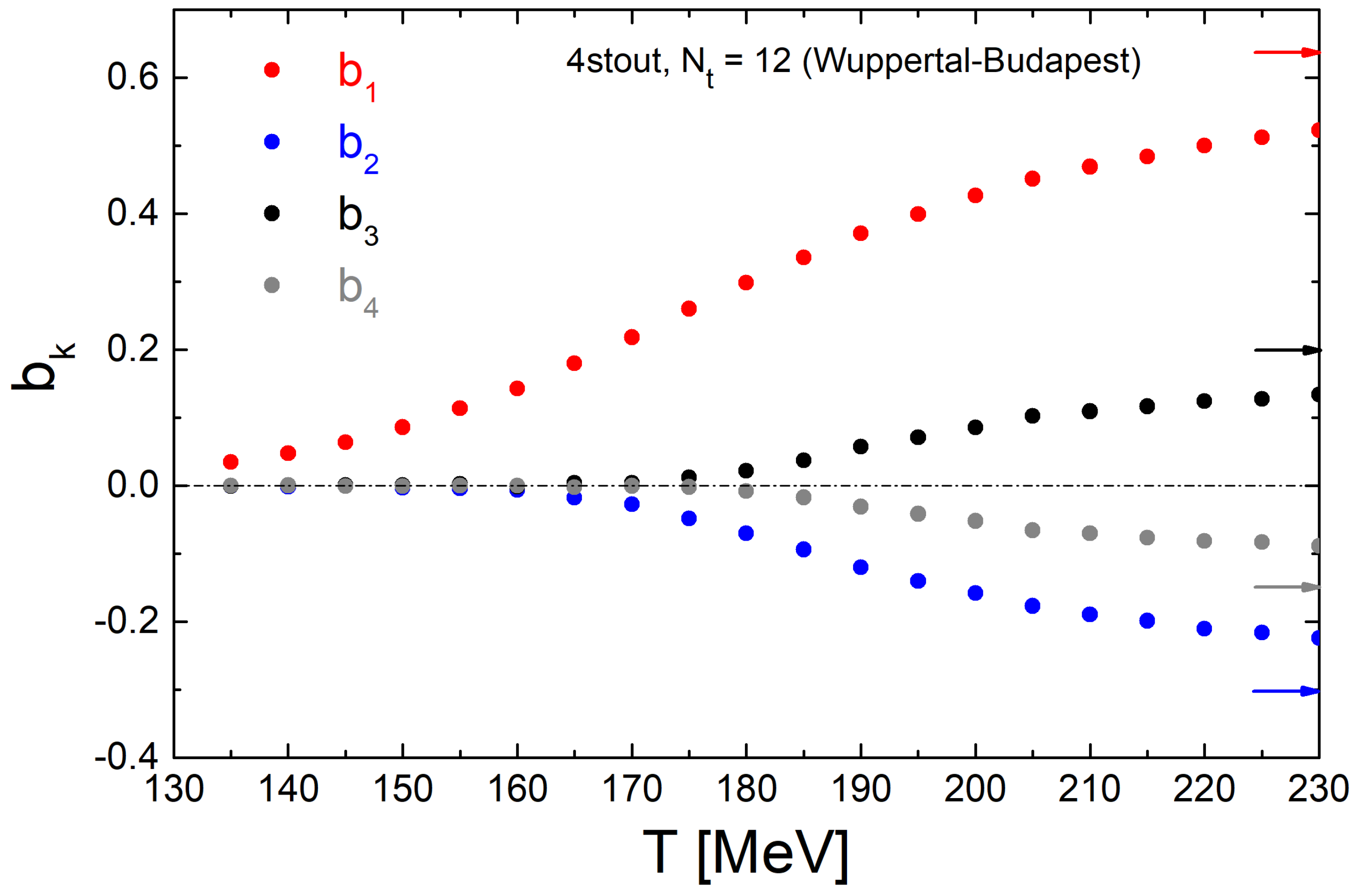
Illustration: TVM parameters fixed to a CP at $T_c = 120$ MeV, $\mu_c = 527$ MeV



Extracted Re[$\mu_{\rm br}$/T]

T [MeV]	Fit to b_1-b_4	True value
150	2.31	2.50
120	4.24	4.39
100	6.11	6.18

Fourier coefficients from lattice



Lattice QCD data (Wuppertal-Budapest), physical quark masses

[**V.V.**, Pasztor, Fodor, Katz, Stoecker, PLB 775, 71 (2017)]

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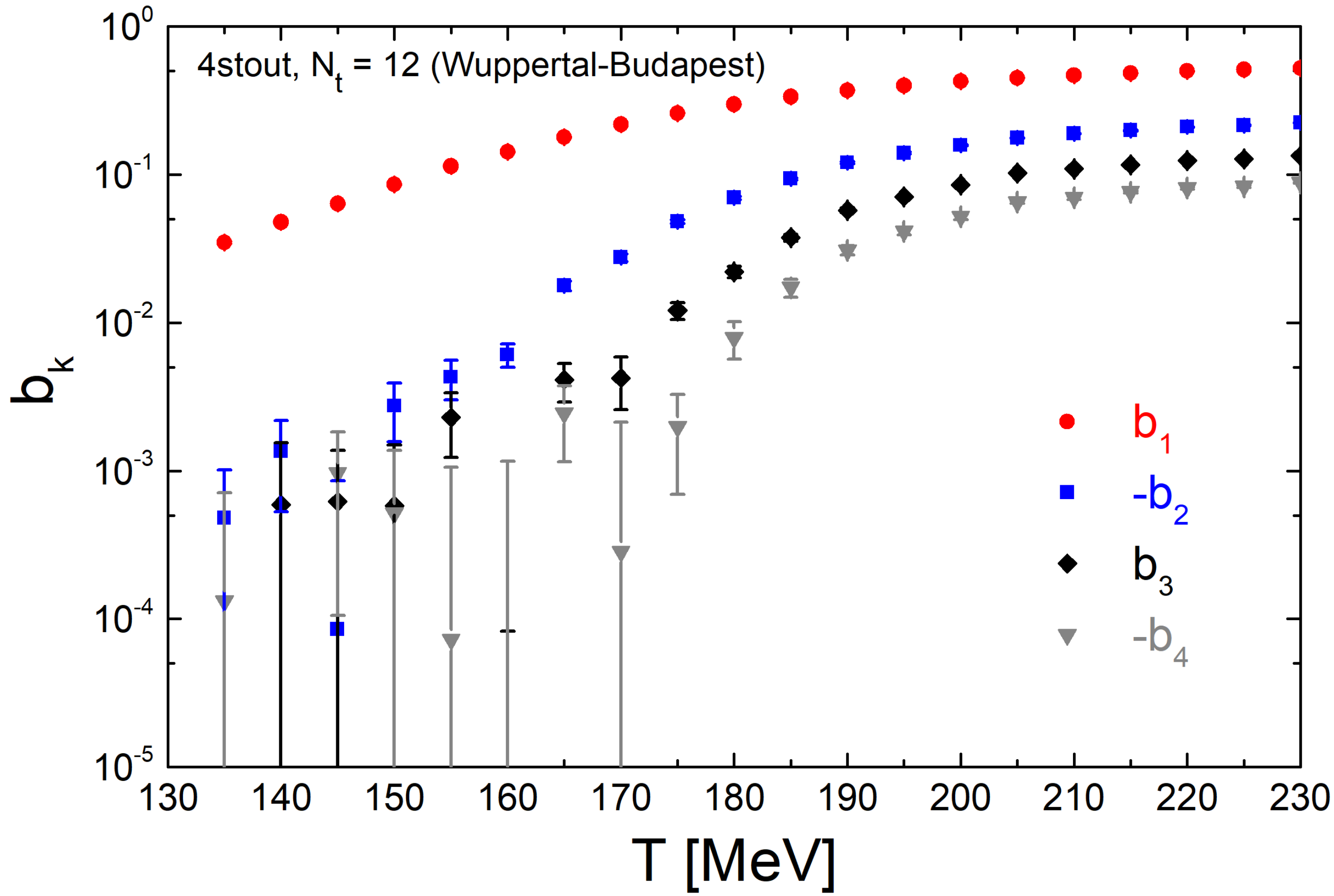

Lattice QCD data (Wuppertal-Budapest), physical quark masses

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Can one extract useful information from lattice data?

Extracting singularities from lattice data

Fit lattice data with an **ansatz:** $\boxed{\log |b_k(T)| = A - \alpha \log k - k \, \mathrm{Re}\left[\frac{\mu_{\mathrm{br}}}{T}\right]}$

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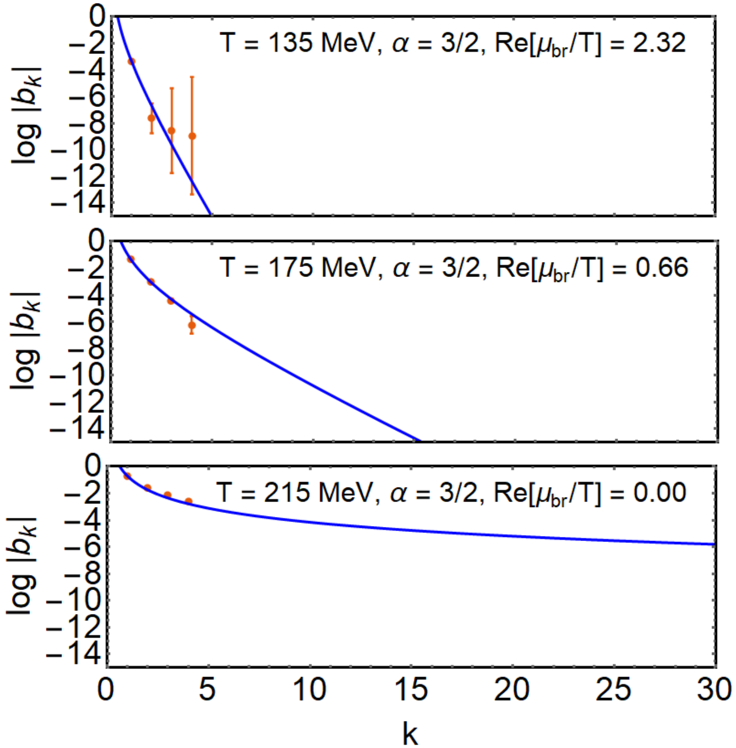

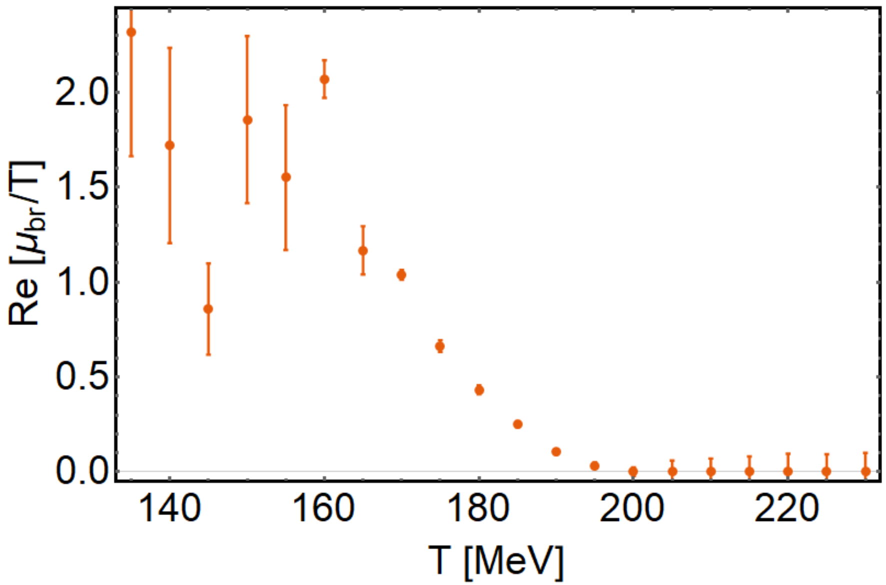

Quite similar results for $1 \le \alpha \le 2$

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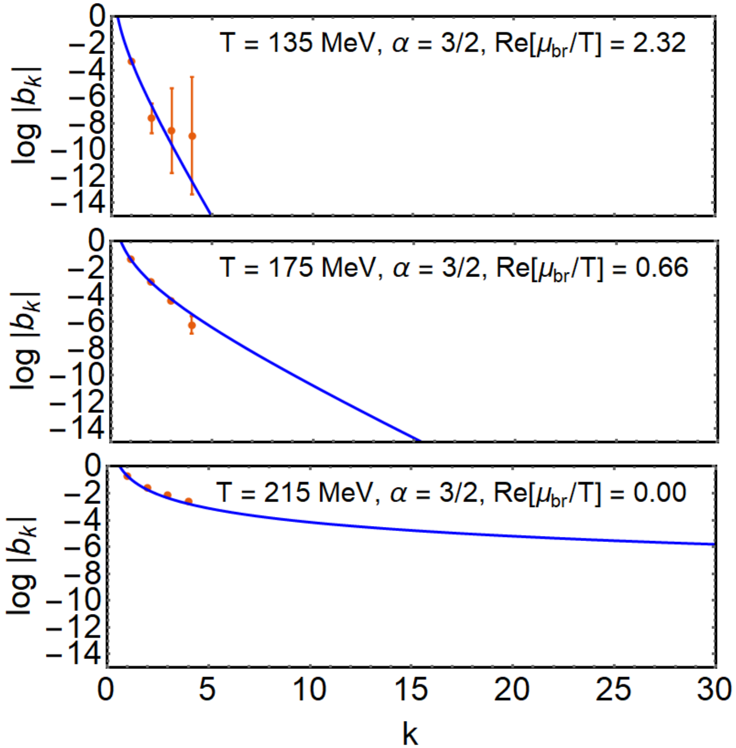

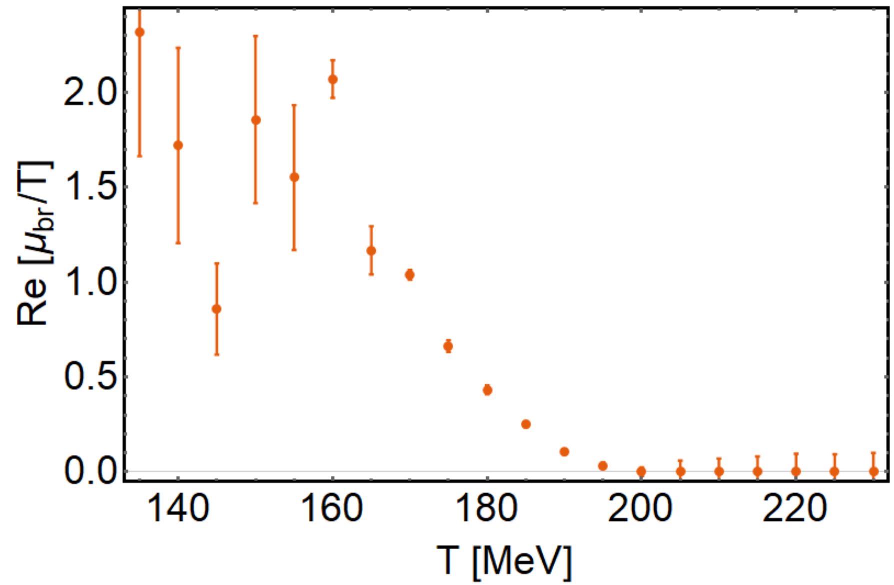

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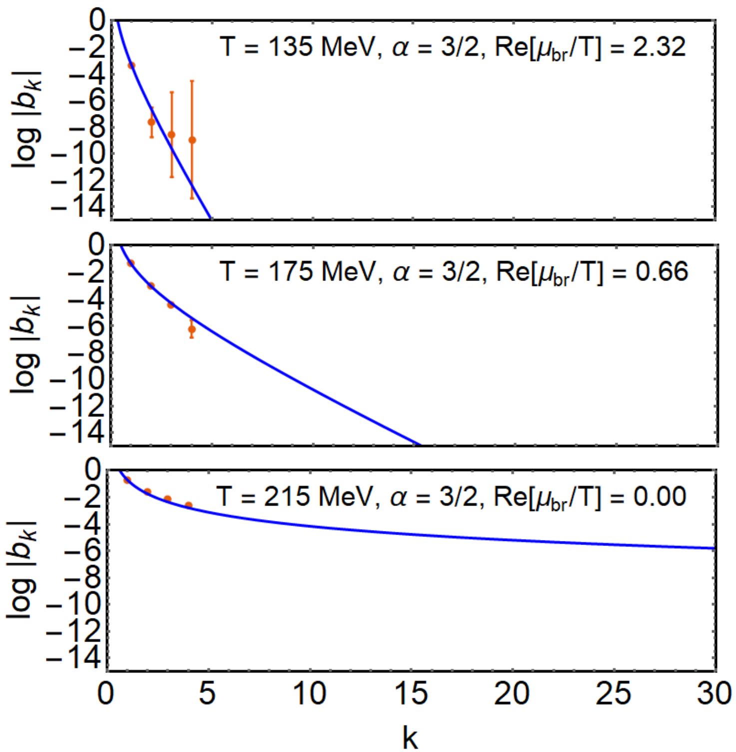

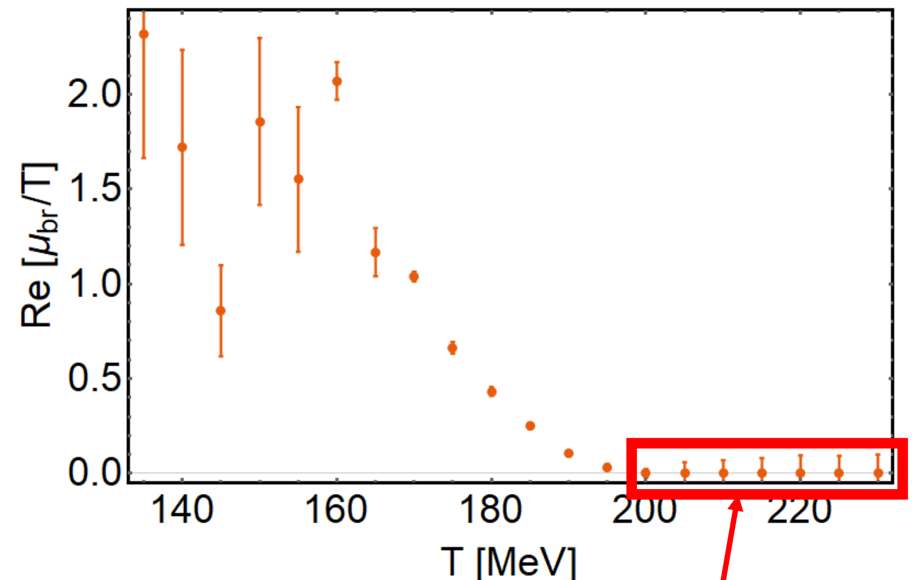

Quite similar results for $1 \leq \alpha \leq 2$

- $b_k \sim$ (-1)$^{k-1}$ in the data $\rightarrow$ $\mathrm{Im}\left[\frac{\mu_{\rm br}}{T}\right] \lesssim \pi$

- $\mathrm{Re}\left[\frac{\mu_{\rm br}}{T}\right] \simeq 0$ for $T \gtrsim 200$ MeV $\rightarrow$ singularity at purely imaginary μ_B

Roberge-Weiss transition?

LQCD: $T_{RW} \sim 208$ MeV [C. Bonati et al., 1602.01426]

Summary

- The "tri-virial" model is an exactly solvable model with a phase transition where Fourier coefficients can worked out explicitly

- Asymptotic behavior associated with a phase transition and a CP

$$T < T_c: \qquad b_k(T) \overset{k\to\infty}{\simeq} A_- \frac{e^{-\frac{k\,\mu_{\mathrm{sp1}}}{T}}}{k^{3/2}}$$

$$T = T_c: \qquad b_k(T) \overset{k\to\infty}{\simeq} A_c \frac{e^{-\frac{k\,\mu_c}{T}}}{k^{4/3}}$$

$$T > T_c: \qquad b_k(T) \overset{k\to\infty}{\simeq} A_+ \frac{e^{-\frac{k\,\mu^R_{\mathrm{br}}}{T}}}{k^{3/2}} \sin\left(k\,\frac{\mu^I_{\mathrm{br}}}{T} + \frac{\theta_0}{2}\right)$$

$+$ *power-law corrections from a difference in universality class from mean-field.*

- Location of thermodynamic singularities can be extracted from LQCD via exponential suppression of Fourier coefficients.
New, accurate data on b_k at T<150 MeV will be useful in the search for (remnants of) critical point/phase transition at finite density.

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Thanks for your attention!

Backup slides

QCD thermodynamics with fugacity expansion

$$\frac{p(T,\mu_B)}{T^4} = \sum_{k=0}^{\infty} p_k(T) \cosh\left(\frac{k\,\mu_B}{T}\right) = \sum_{k=-\infty}^{\infty} \tilde{p}_{|k|}(T)\, e^{k\mu_B/T}$$

No sign problem on the lattice at imaginary $\mu_B \to i\tilde{\mu}_B$

Observables obtain trigonometric Fourier series form

Baryon density: $\frac{\rho_B(T,\mathrm{i}\tilde{\mu}_B)}{T^3} = \mathrm{i} \sum_{k=1}^{\infty} b_k(T) \sin\left(\frac{k\,\tilde{\mu}_B}{T}\right), \quad b_k(T) \equiv k\, p_k(T)$

$$b_k(T) = \frac{2}{\pi\, T^4} \int_0^{\pi T} d\tilde{\mu}_B\, [\mathrm{Im}\, \rho_B(T, i\tilde{\mu}_B)] \sin(k\,\tilde{\mu}_B/T)$$

Ideal (Boltzmann) HRG:

$$\frac{\rho_B}{T^3} = b_1(T) \sinh\left(\frac{\mu_B}{T}\right)$$

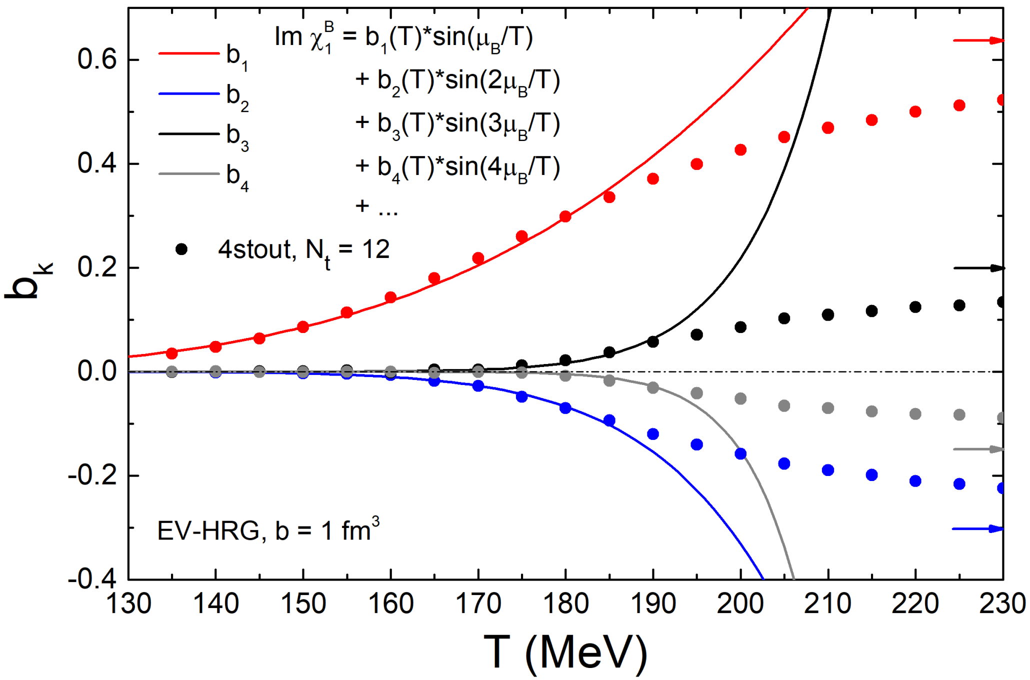
Massless quarks (Stefan-Boltzmann limit):

$$b_k^{\mathrm{SB}} = \frac{(-1)^{k+1}}{k} \frac{4\,[3 + 4\,(\pi\, k)^2]}{27\,(\pi\, k)^2}$$

HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) $V \to V - bN$

[Hagedorn, Rafelski, '80; Dixit, Karsch, Satz, '81; Cleymans et al., '86; Rischke et al., Z. Phys. C '91]



HRG with baryonic EV:

$$p_B(T,\mu_B) = p_B^{\mathrm{id}}(T,\mu_B - b p_B)$$

$$b_k^{ev}(T) = (-1)^{k-1}\,\frac{2\,k^k}{k!}\,(b\,T^3)^{k-1}\left[\frac{\phi_B(T)}{T^3}\right]^k$$

V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852

- Non-zero $b_k(T)$ for $k \geq 2$ signal deviation from ideal HRG
- EV interactions between baryons ($b \approx 1$ fm^3) reproduce lattice trend

Cluster Expansion Model (CEM)

Model formulation:

- Fugacity expansion for baryon number density

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T)$$

- $b_1(T)$ and $b_2(T)$ are model input
- All higher order coefficients are predicted: $b_k(T) = \alpha_k^{SB} \frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$

Physical picture: Hadron gas with repulsion at moderate T, "weakly" interacting quarks and gluons at high T

Summed analytic form:

$$\frac{\rho_B(T,\mu_B)}{T^3} = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_1(x_+) - \mathrm{Li}_1(x_-)\right] + 3\left[\mathrm{Li}_3(x_+) - \mathrm{Li}_3(x_-)\right]\right\}$$

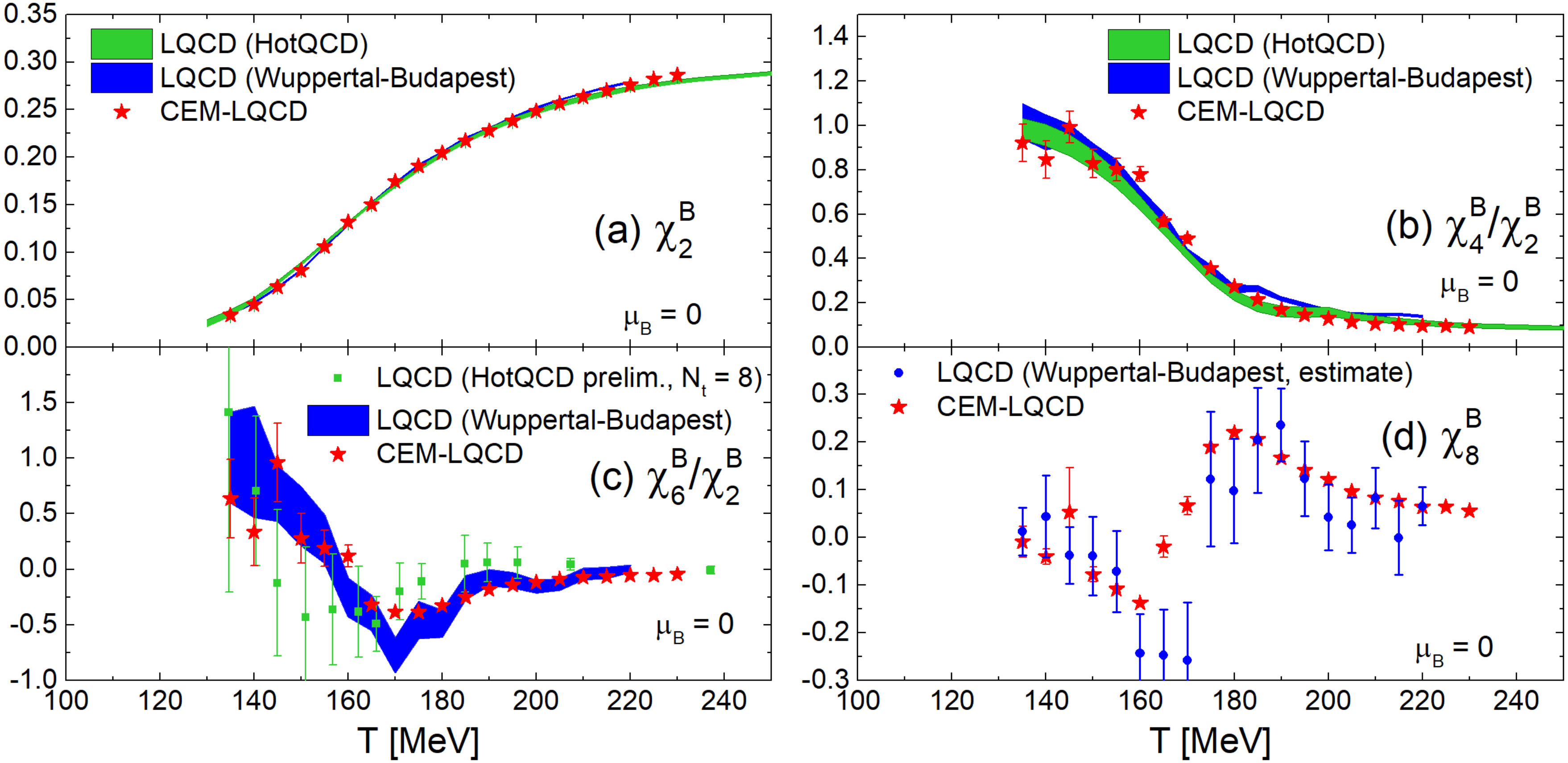
$$\hat{b}_{1,2} = \frac{b_{1,2}(T)}{b_{1,2}^{\mathrm{SB}}}, \quad x_\pm = -\frac{\hat{b}_2}{\hat{b}_1} e^{\pm\mu_B/T}, \quad \mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$$

Regular behavior at real μ_B $\rightarrow$ *no-critical-point scenario*

CEM: Baryon number susceptibilities

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2\text{-k}}(x_+) + (-1)^k\,\mathrm{Li}_{2\text{-k}}(x_-)\right] + 3\left[\mathrm{Li}_{4\text{-k}}(x_+) + (-1)^k\,\mathrm{Li}_{4\text{-k}}(x_-)\right]\right\}$$

CEM-LQCD: $b_1(T)$ and $b_2(T)$ from LQCD simulations at imaginary μ_B

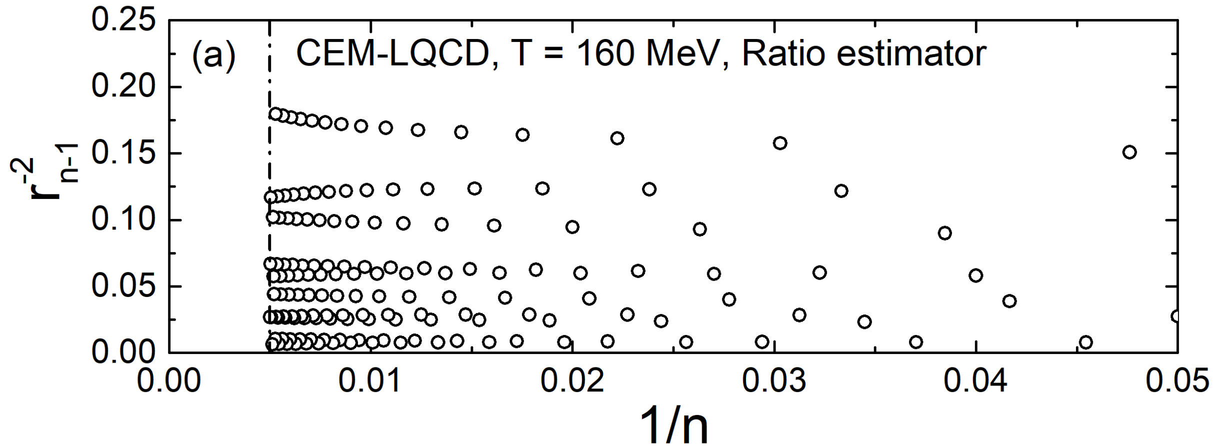


Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)

Using estimators for radius of convergence

a) Ratio estimator:

$$r_n = \left| \frac{(2n+2)(2n+1)\chi^B_{2n}}{\chi^B_{2n+2}} \right|^{1/2}$$

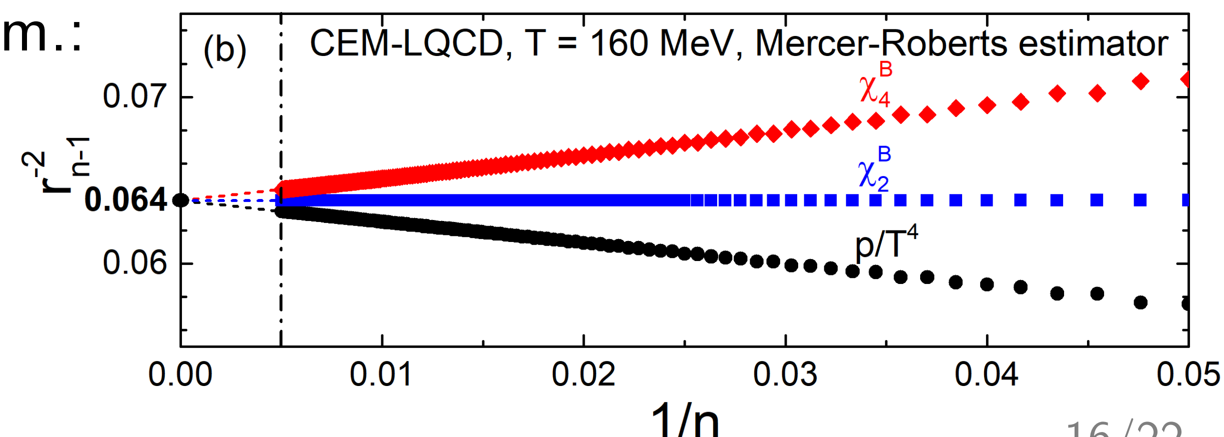


Ratio estimator is *unable* to determine the radius of convergence,
nor to provide an upper or lower bound, *so use it with care!!*

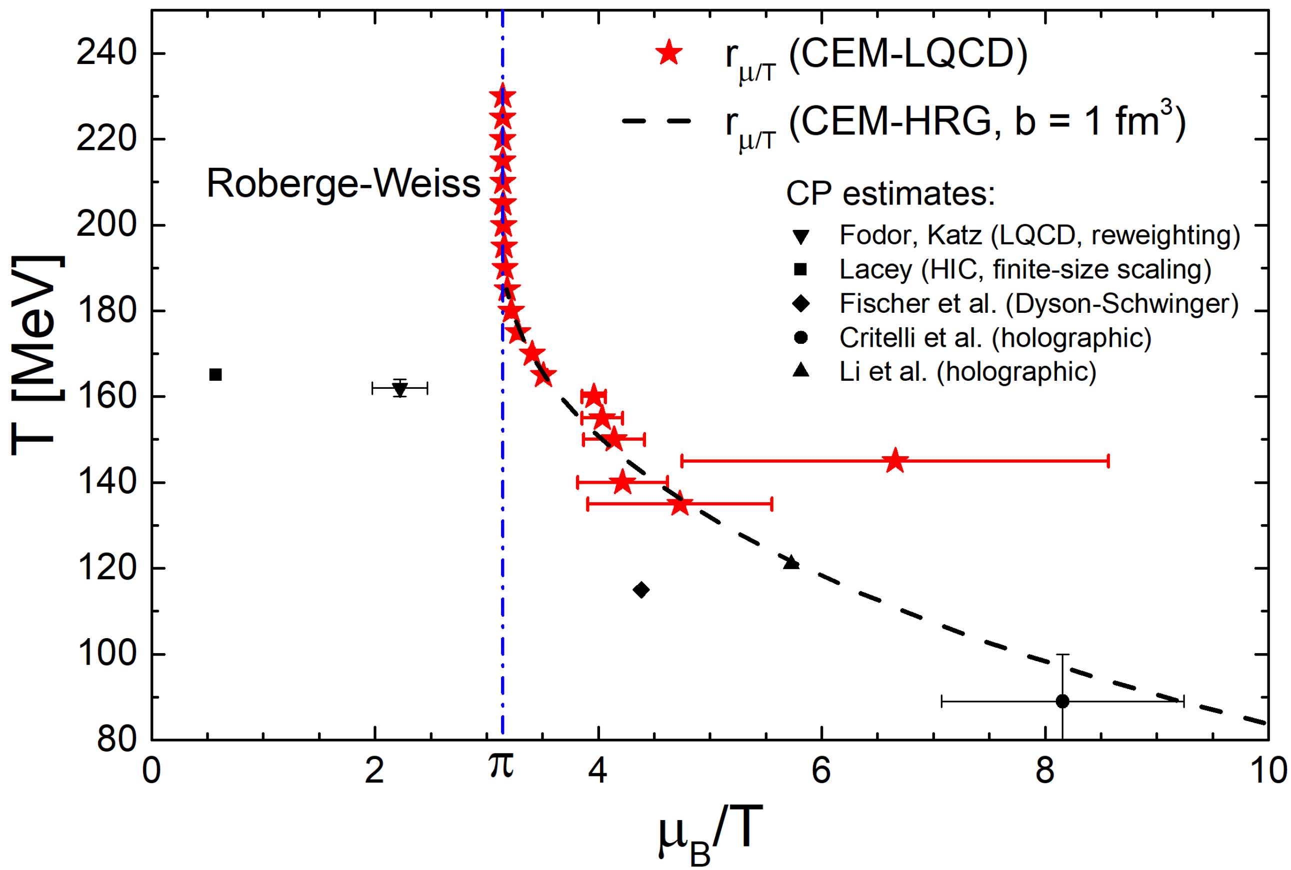
b) Mercer-Roberts estim.:

$$r_n = \left| \frac{c_{n+1}\, c_{n-1} - c_n^2}{c_{n+2}\, c_n - c_{n+1}^2} \right|^{1/4}$$

$$c_n = \frac{\chi^B_{2n}}{(2n)!}$$



CEM: Radius of convergence



Radius of convergence approaches Roberge-Weiss transition value

- At $T > T_{RW}$ expected $\left[\frac{\mu_B}{T}\right]_c = \pm i\pi$ [Roberge, Weiss, NPB '86] $T_{RW} \sim 208$ MeV [C. Bonati et al., 1602.01426]
- Complex plane singularities interfere with the search for CP

Expected asymptotics

- At low T/densities QCD $\simeq$ ideal hadron resonance gas

$$\frac{p^{\rm hrg}(T,\mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2\,\frac{\phi_B(T)}{T^3}\cosh\left(\frac{\mu_B}{T}\right),$$

$$\phi_B(T) = \sum_{i\in B}\int dm\,\rho_i(m)\,\frac{d_i\,m^2\,T}{2\pi^2}\,K_2\left(\frac{m}{T}\right),$$

$$p_0^{hrg}(T) = \frac{\phi_M(T)}{T^3},\quad p_1^{hrg}(T) = \frac{2\,\phi_B(T)}{T^3},\quad p_k^{\rm hrg}(T) \equiv 0,\ k \geq 2$$

- At high T QCD $\simeq$ ideal gas of massless quarks and gluons

$$\frac{p^{\rm SB}(T,\mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s}\left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_B}{3T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_B}{3T}\right)^4\right],$$

$$p_0^{\rm SB} = \frac{64\pi^2}{135},\quad p_k^{\rm SB} = \frac{(-1)^{k+1}}{k^2}\,\frac{4\,[3+4\,(\pi k)^2]}{27\,(\pi k)^2},\quad b_k^{SB} = k\,p_k^{\rm SB}.$$

Lattice data explore intermediate, transition region $130 < T < 230$ *MeV*

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*In this study we assume that $\mu_S = \mu_Q = 0$