Non-Gaussian moments of fluctuations of conserved charges:

Applications for strongly interacting matter

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- Search for the critical point of QCD with non-Gaussian fluctuations
- Lattice QCD data on fluctuations: EoS at finite μ and role of hadronic interactions

Thanks to: D. Anchishkin, M. Gorenstein, R. Poberezhnyuk, H. Stoecker

10th BGL Conference, Gyöngyös, Hungary

August 24, 2017

Measures of non-Gaussian fluctuations

Higher-order moments of fluctuations

Let N be a random variable and $P(N)$ its probability distribution.

k-th moment: $$\langle N^k \rangle = \sum_N N^k P(N)$$

k-th cumulant: $$\kappa_k : \quad \log\langle e^{tN}\rangle = \sum_{n=1}^{\infty} \kappa_n \frac{t^n}{n!}$$

Mean: $$M = \langle N \rangle$$

Variance: $$\sigma^2 = \langle (\Delta N)^2 \rangle = \langle (N - \langle N \rangle)^2 \rangle$$

Scaled variance: $$\frac{\sigma^2}{M} = \frac{\kappa_2}{\kappa_1} = \frac{\sigma^2}{\langle N \rangle}$$ width

(Scaled) **skewness:** $$S\sigma = \frac{\kappa_3}{\kappa_2} = \frac{\langle (\Delta N)^3 \rangle}{\sigma^2}$$ asymmetry

(Scaled) **kurtosis:** $$\kappa\sigma^2 = \frac{\kappa_4}{\kappa_2} = \frac{\langle (\Delta N)^4 \rangle - 3\,\langle (\Delta N)^2 \rangle^2}{\sigma^2}$$ peakedness

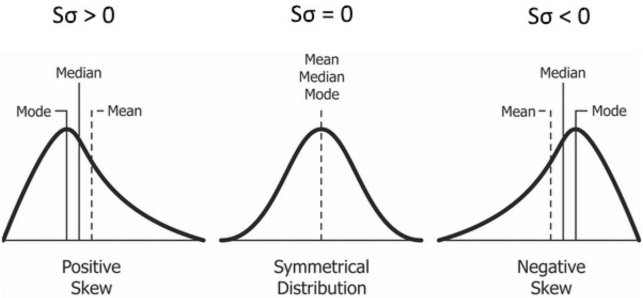
and so on...

In heavy-ion collisions/Lattice QCD N can be conserved charge (baryon, electric, strangeness) or some particle number in a specific phase-space region

Non-Gaussian fluctuations: Skewness

(Normalized) skewness measures the degree of asymmetry of distribution

$$S\sigma \; = \; \frac{\kappa_3}{\kappa_2} = \frac{\langle(\Delta N)^3\rangle}{\sigma^2} \; .$$



Baselines:

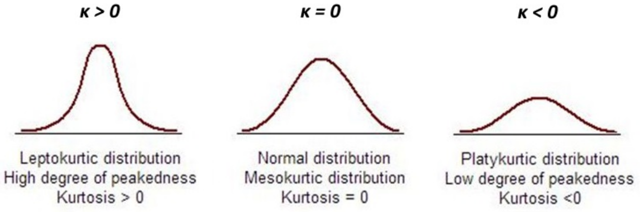
- Gaussian: $S\sigma = 0$
- Poisson: $S\sigma = 1 \quad \leftarrow \quad$ ideal Boltzmann gas in grand canonical ensemble

Deviations from zero signal the non-Gaussian shape of distribution

Non-Gaussian fluctuations: Kurtosis

(Normalized) kurtosis measures "peakedness" of distribution

$$\kappa\sigma^2 = \frac{\kappa_4}{\kappa_2} = \frac{\langle(\Delta N)^4\rangle - 3\langle(\Delta N)^2\rangle^2}{\sigma^2} .$$



Baselines:

- Gaussian: $\kappa\sigma^2 = 0$
- Poisson: $\kappa\sigma^2 = 1 \quad \leftarrow \quad$ ideal Boltzmann gas in grand canonical ensemble

Deviations from zero signal the non-Gaussian shape of the distribution

Fluctuations in thermodynamic systems

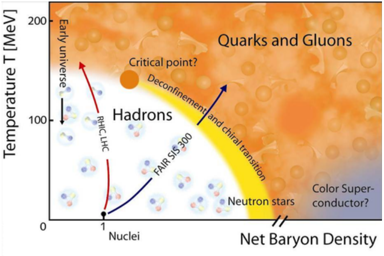
Why are fluctuations interesting?

In thermodynamics fluctuations are related to susceptibilities $\chi^{(n)}$

$$\chi^{(n)} = \frac{\partial^n (p/T^4)}{\partial (\mu/T)^n}$$

$$\frac{\sigma^2}{M} = \frac{\chi^{(2)}}{\chi^{(1)}}, \qquad S\sigma = \frac{\chi^{(3)}}{\chi^{(2)}}, \qquad \kappa\sigma^2 = \frac{\chi^{(4)}}{\chi^{(2)}},$$

Fluctuations are very sensitive to QCD equation of state and they can be used to study QCD phase transitions



Near CP $\sim$ increasing powers of ξ*

$$\chi^{(2)} \sim \xi^2$$
$$\chi^{(3)} \sim \xi^{4.5}$$
$$\chi^{(4)} \sim \xi^7$$

Infinite system: $\xi \to \infty$ at CP

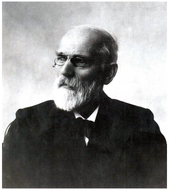
In HIC $\xi \lesssim 2-3$ fm

*M. Stephanov, Phys. Rev. Lett. **102**, 032301 (2009)

Non-Gaussian fluctuation signals of a critical point: analysis within the van der Waals model

van der Waals (VDW) equation

$$P(T,V,N) = \frac{NT}{V-bN} - a\frac{N^2}{V^2}$$



Formulated in 1873.

Simplest model which contains attractive and repulsive interactions

Contains 1st order phase transition and critical point

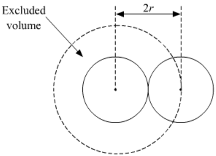
Can elucidate role of fluctuations in phase transitions

Nobel Prize in 1910.

Two ingredients:

1) Short-range repulsion: excluded volume (EV) procedure,

$V \to V - bN, \qquad b = 4\frac{4\pi r^3}{3}$



2) Intermediate range attraction in mean-field approximation,

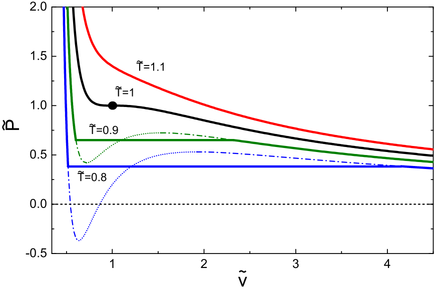
$P \to P - a\,n^2$

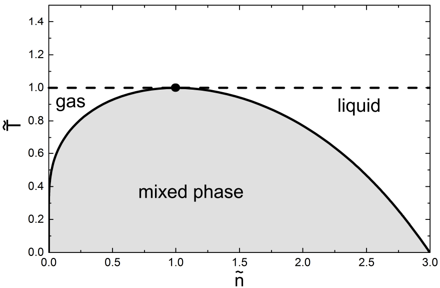
Motivation:

- Toy model to study fluctuations near critical point
- Include essential features of nuclear matter physics

van der Waals equation

- vdW isotherms show irregular behavior below certain temperature T_C
- Below T_C isotherms are corrected by Maxwell's rule of equal areas
- Results in appearance of mixed phase





Reduced variables

$$\tilde{p} = \frac{p}{p_C},\ \tilde{n} = \frac{n}{n_C},\ \tilde{T} = \frac{T}{T_C}$$

Critical point

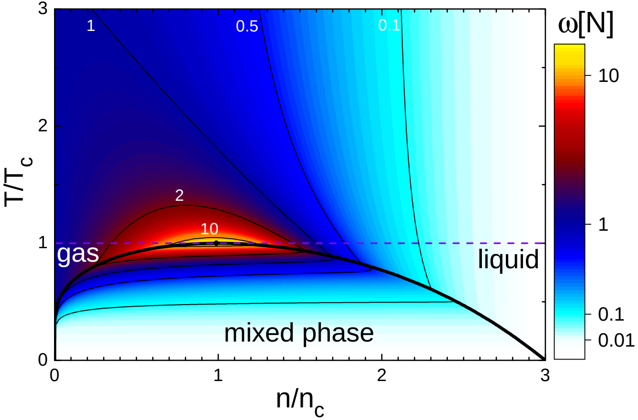
$$\frac{\partial p}{\partial v} = 0, \quad \frac{\partial^2 p}{\partial v^2} = 0, \quad v = V/N$$

$$p_C = \frac{a}{27b^2},\ n_C = \frac{1}{3b},\ T_C = \frac{8a}{27b}$$

Scaled variance for classical vdW equation

Particle number fluctuations in classical vdW gas within GCE

$$\omega[N] = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} = \frac{\chi_2}{\chi_1} = \left[\frac{1}{(1-bn)^2} - \frac{2an}{T} \right]^{-1}$$

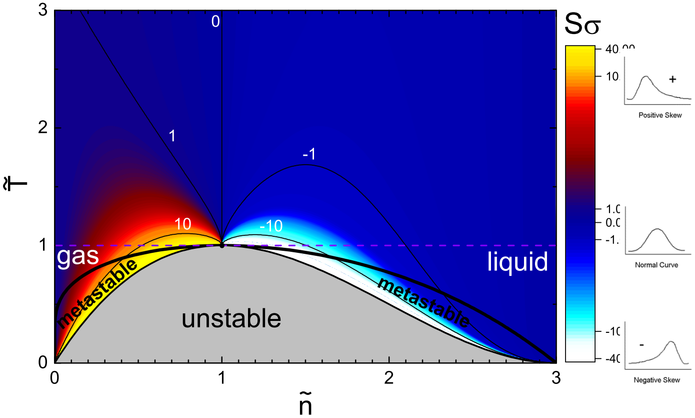


- Repulsive interactions suppress N-fluctuations
- Attractive interactions enhance N-fluctuations

V.V., Anchishkin, Gorenstein, J. Phys. A 48, 305001 (2015)

Classical vdW equation: Skewness

Skewness: $S\sigma \;=\; \dfrac{\langle(\Delta N)^3\rangle}{\sigma^2} \;=\omega[N]+\dfrac{T}{\omega[N]}\left(\dfrac{\partial\omega[N]}{\partial\mu}\right)_T$ asymmetry



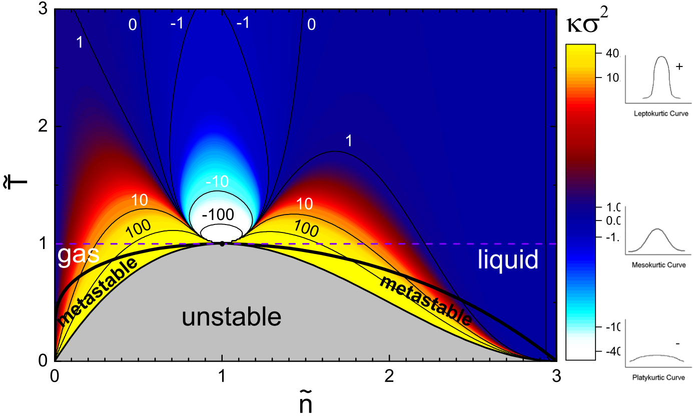
Skewness is

- Positive (right-tailed) in gaseous phase
- Negative (left-tailed) in liquid phase

V.V., R. Poberezhnyuk, D. Anchishkin, M. Gorenstein, J. Phys. A 49, 015003 (2016)

Classical vdW equation: Kurtosis

Kurtosis: $\kappa\sigma^2 = \dfrac{\langle(\Delta N)^4\rangle - 3\langle(\Delta N)^2\rangle^2}{\sigma^2}$ peakedness



Kurtosis is negative (flat) above critical point (crossover), positive (peaked) elsewhere and very sensitive to the proximity of the critical point

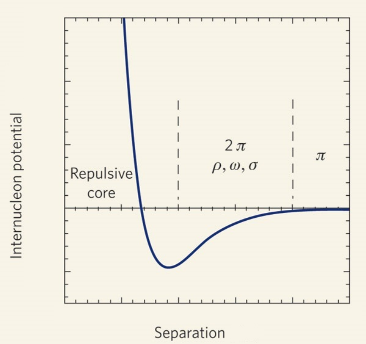
V.V., R. Poberezhnyuk, D. Anchishkin, M. Gorenstein, J. Phys. A 49, 015003 (2016)

Nucleon-nucleon interaction

A more specific application: nuclear matter

Nucleon-nucleon potential:

- Repulsive core at small distances
- Attraction at intermediate distances
- No resonance structure
- Suggestive similarity to vdW interactions
- Could nuclear matter be described by vdW equation?



Nuclear matter with quantum van der Waals (QvdW) equation

$$p(T,n) = p_q^{\mathrm{id}}\left(T, \frac{n}{1-bn}\right) - a\,n^2$$

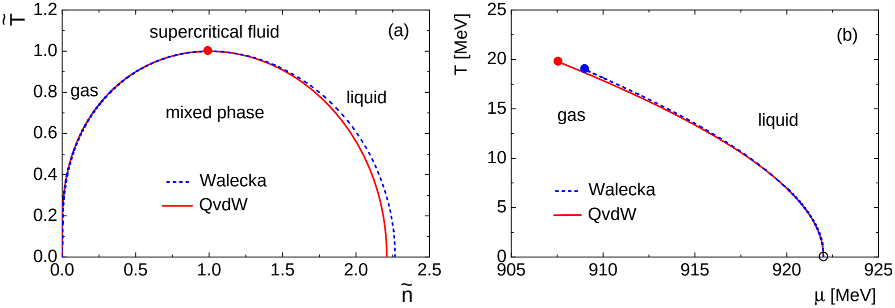
$E/A = -16$ MeV, $n_0 = 0.16$ fm^{-3} $\Rightarrow$ $a_{NN} = 329$ MeV fm^3, $b_{NN} = 3.42$ fm^3

V.V., D. Anchishkin, M. Gorenstein, PRC '15; K. Redlich, K. Zalewski, APPB '16.
Details about the mean-field approach in the talk of D. Anchishkin

QvdW gas of nucleons: phase diagram

Model predicts critical point at $T_c \cong 19.7$ MeV and $n_c \cong 0.07$ fm^{-3}

Experimental estimate[1]: $T_c = 17.9 \pm 0.4$ MeV, $n_c = 0.06 \pm 0.01$ fm^{-3}



Despite conceptual differences, QvdW results are very similar to Walecka model[2]

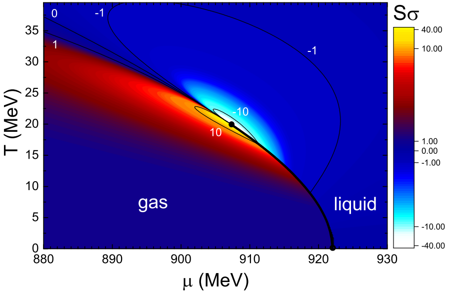
[1] J.B. Elliot, P.T. Lake, L.G. Moretto, L. Phair, Phys. Rev. C 87, 054622 (2013)

[2] R. Poberezhnyuk, V.V., D. Anchishkin, M. Gorenstein, 1708.05605

QvdW gas of nucleons: skewness and kurtosis

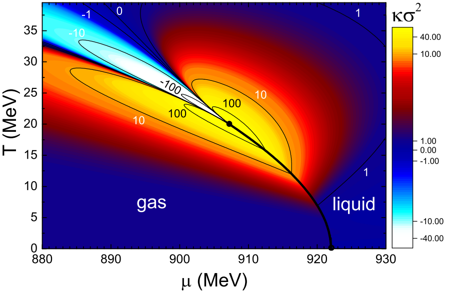
Skewness

$$S\sigma = \omega[N] + \frac{T}{\omega[N]} \left(\frac{\partial \omega[N]}{\partial \mu} \right)_T$$



Kurtosis

$$\kappa\sigma^2 = (S\sigma)^2 + T \left(\frac{\partial [S\sigma]}{\partial \mu} \right)_T$$

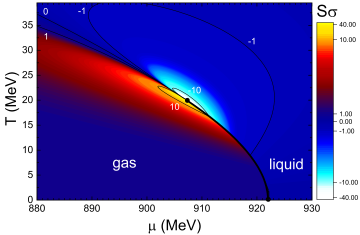


- Structure of fluctuations is consistent with the classical VDW model predictions
- Expected to hold true for QCD CP due to universality of critical behavior
- Non-trivial behavior can be probed by heavy-ion collision experiments

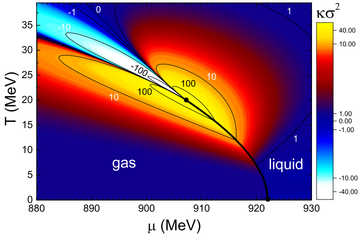
V.V., D. Anchishkin, M. Gorenstein, R. Poberezhnyuk, PRC 91, 064314 (2015)

QvdW gas of nucleons: skewness and kurtosis

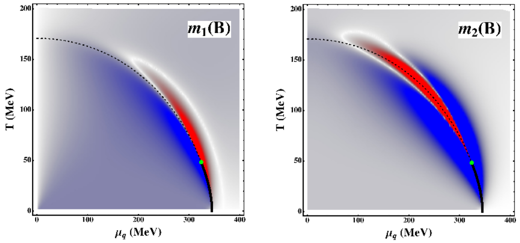
QvdW Skewness



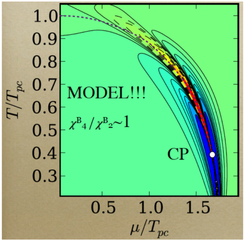
QvdW Kurtosis



NJL, J.W. Chen et al., PRD 93, 034037 (2016)



PQM, V. Skokov, QM2012



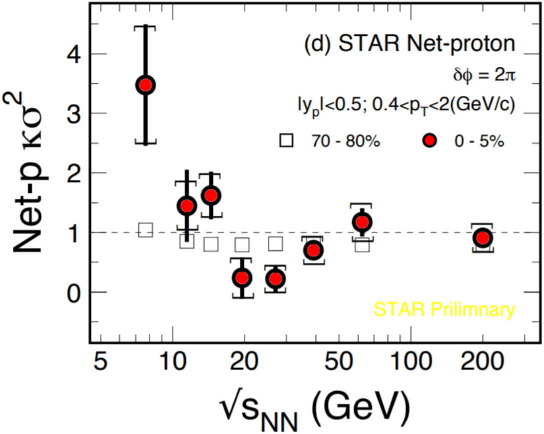
Fluctuation patterns in QvdW very similar to effective QCD models

Search for CP in heavy-ion collision experiments

Experimental search for QCD CP using non-Gaussian fluctuations is underway

Measurements at STAR and NA61/SHINE experiments

X. Luo, CPOD2014, QM2015



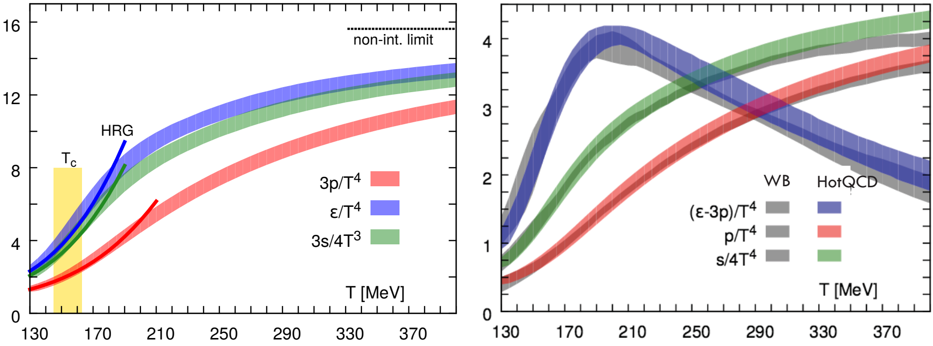
Interpretation challenging, many “background” things contribute

No definitive conclusions regarding the location or existence of QCD CP yet

Conserved charges fluctuations in the context of Lattice QCD

QCD equation of state at $\mu = 0$

Lattice QCD is a first-principle tool which provides equation of state at $\mu_B = 0$[1]



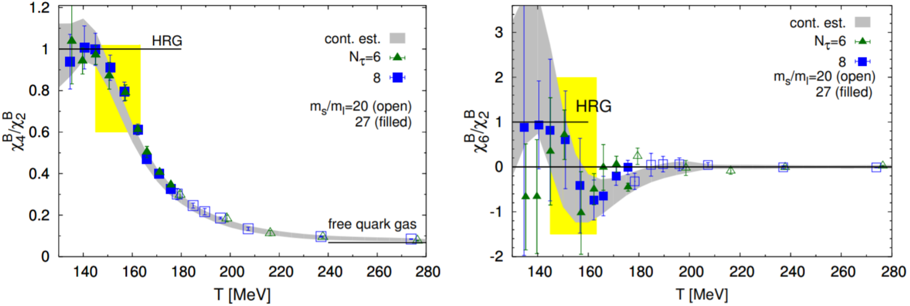
- QCD exhibits crossover-type transition at $T \sim 140 - 190$ MeV
- No actual phase transition at $\mu = 0$
- Basic thermodynamic quantities described well by ideal hadron resonance gas model up to temperatures beyond crossover

Bazavov et al. [HotQCD Collaboration], PRD 90, 094503 (2014)
Borsanyi et al. [Wuppertal-Budapest Collaboration], PLB 730, 99 (2014)

Lattice QCD susceptibilities at $\mu = 0$

Susceptibilities of the conserved charges at $\mu = 0$ are calculated as well:

$$\chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l \, \partial(\mu_S/T)^m \, \partial(\mu_Q/T)^n}$$



- Fluctuations probe finer details of the EoS
- Ideal HRG breaks down rapidly at $T \sim 150$ MeV
- This deviation commonly attributed to onset of deconfinement

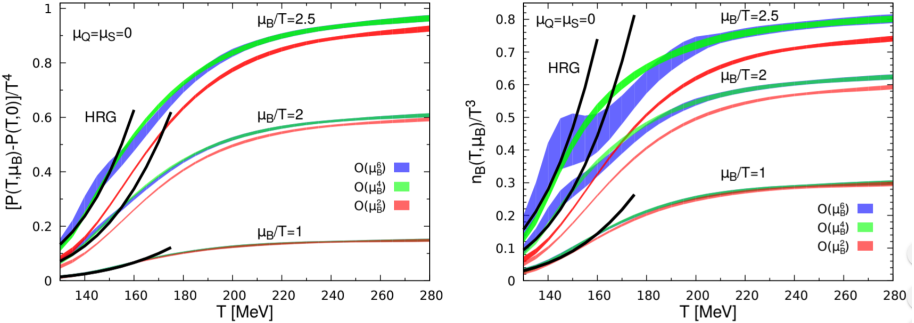
Bazavov et al. [BNL-Bielefeld-CCNU Collaboration], PRD 95, 054504 (2017)

QCD equation of state at small but finite μ

Lattice QCD simulations are restricted to $\mu = 0$ region due to sign problem

(Non-Gaussian) fluctuations at $\mu = 0$ provide EoS at finite μ via Taylor expansion:

$$p(T, \mu_B) = p(T, 0) + \frac{\chi_2^B(T,0)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T,0)}{4!}(\mu_B/T)^4 + \ldots$$



- QCD EoS now available up to $\mu_B/T \simeq 2$ via Taylor expansion
- Deviations from ideal HRG set in earlier at larger μ_B/T
- But what is the role of hadronic interactions beyond ideal HRG?

Bazavov et al. [BNL-Bielefeld-CCNU Collaboration], PRD 95, 054504 (2017)

van der Waals interactions in hadron resonance gas

Ideal HRG includes attractive interactions on the level of resonance formation

It misses the vdW nature of baryon-baryon interactions

(Q)vdW-HRG model

- Identical vdW interaction terms between all baryons added
- Baryon-antibaryon, meson-meson, meson-baryon vdW terms not considered
- Baryon vdW parameters extracted from ground state of nuclear matter ($a = 329$ MeV fm^3, $b = 3.42$ fm^3)

Three independent subsystems: mesons + baryons + antibaryons

$$p(T, \boldsymbol{\mu}) = P_M(T, \boldsymbol{\mu}) + P_B(T, \boldsymbol{\mu}) + P_{\bar{B}}(T, \boldsymbol{\mu}),$$

Ideal gas of mesons: $$P_M(T, \boldsymbol{\mu}) = \sum_{j \in M} p_j^{\rm id}(T, \mu_j)$$

QvdW gas of baryons: $$P_B(T, \boldsymbol{\mu}) = \sum_{j \in B} p_j^{\rm id}(T, \mu_j^{B*}) - a\, n_B^2 \simeq \frac{T\, n_B}{1 - b\, n_B} - a\, n_B^2$$

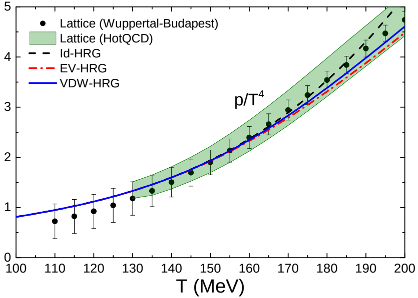
In this simplest setup model is essentially "parameter-free"

V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)

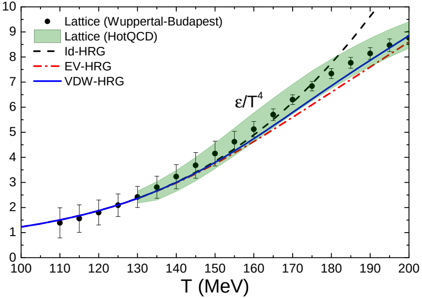
vdW-HRG at $\mu_B = 0$: thermodynamic functions

Comparison of vdW-HRG with lattice QCD at $\mu_B = 0$

Pressure p/T^4



Energy density ε/T^4



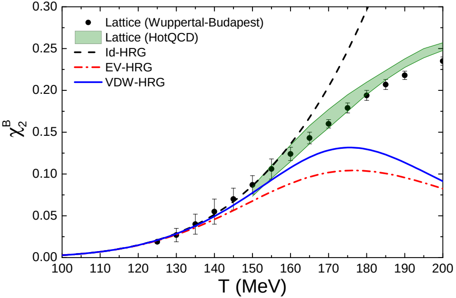
- vdW-HRG does not spoil existing agreement of Id-HRG with LQCD despite significant excluded-volume interactions between baryons
- Not surprising: matter meson-dominated at $\mu_B = 0$

V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)

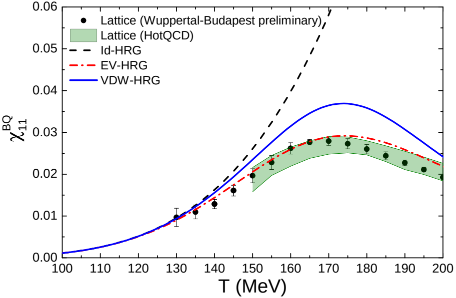
vdW-HRG at $\mu_B = 0$: baryon number fluctuations

Susceptibilities: $\chi_{lmn}^{BSQ} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l \, \partial(\mu_S/T)^m \, \partial(\mu_Q/T)^n}$

Net-baryon χ_2^B



Baryon-charge correlator χ_{11}^{BQ}



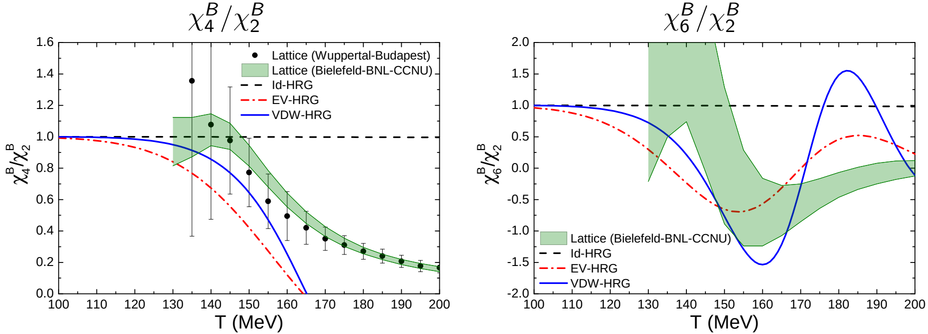
- Very different qualitative behavior between Id-HRG and vdW-HRG
- For χ_2^B lattice data is between Id-HRG and vdW-HRG at high T
- For χ_{11}^{BQ} lattice data is below all models, closer to EV-HRG

V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)

vdW-HRG at $\mu_B = 0$: baryon number fluctuations

Higher-order fluctuations are expected to be even more sensitive

To leading order: $(\chi_2^B - \chi_4^B) \propto 24 b\, n_B^2$ and $(\chi_2^B - \chi_6^B) \propto 120 b\, n_B^2$



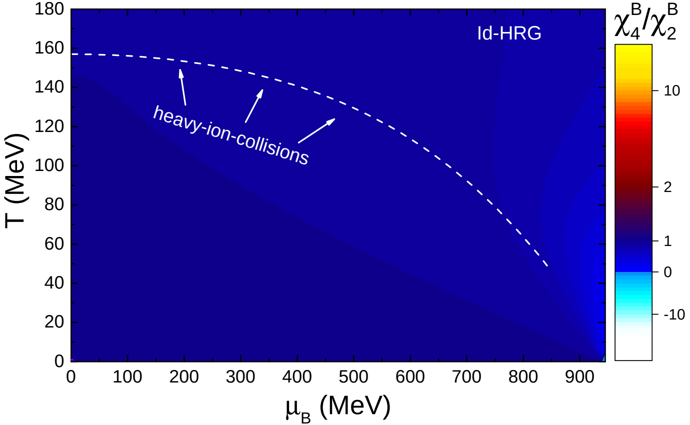
- χ_4^B deviates from χ_2^B at high enough T, they stay equal in Id-HRG
- vdW-HRG predicts strong non-monotonic behavior for χ_6^B/χ_2^B
- Onset of deviations of LQCD results from ideal HRG can be understood in terms of vdW-like interactions between baryons*

V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)

*These conclusions were also corroborated in P. Huovinen, P. Petreczky, 1708.00879

vdW-HRG at finite μ_B

Net-baryon fluctuations in T-μ_B plane: χ_4^B/χ_2^B

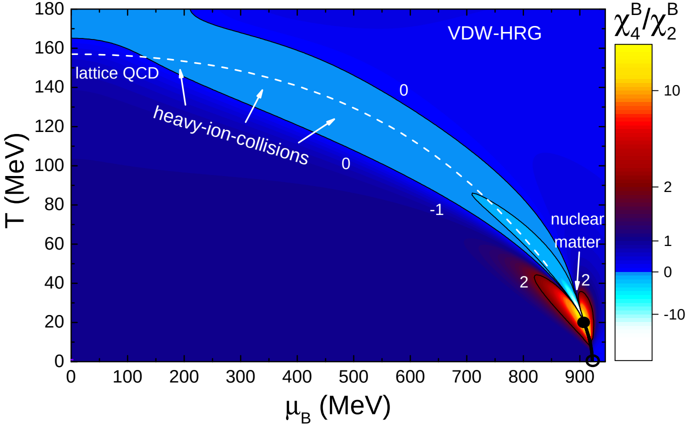


- Almost no effect in Id-HRG, only Fermi statistics...

V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)
see also A. Mukherjee, J. Steinheimer, S. Schramm, Phys. Rev. C 96, 025205 (2017)

vdW-HRG at finite μ_B

Net-baryon fluctuations in T-μ_B plane: χ_4^B/χ_2^B



- Almost no effect in Id-HRG, only Fermi statistics...
- Rather rich structure for vdW-HRG, huge effect of vdW interactions!
- Nuclear liquid-gas criticality shines brightly across whole phase diagram

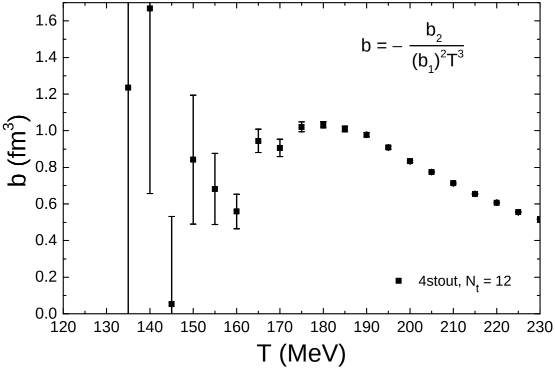
V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)
see also A. Mukherjee, J. Steinheimer, S. Schramm, Phys. Rev. C 96, 025205 (2017)

Repulsive baryonic interactions from imaginary μ_B LQCD data

Lattice QCD is free of sign problem at imaginary μ

E.g., net-baryon density is imaginary and has trigonometric series form

$$\mu_B \to i\tilde{\mu}_B \quad \Rightarrow \quad \frac{n_B(T, i\tilde{\mu}_B)}{T^3} = i \sum_{j=1}^{\infty} b_j(T) \sin(j\,\tilde{\mu}_B/T)$$



$b_j \propto$ cluster integrals

2nd virial coefficient:

$$b(T) = -\frac{b_2(T)}{[b_1(T)]^2\, T^3}$$

- $b(T)$ mostly consistent with 1 fm^3 at $T < 190$ MeV
- $b \sim 1/T^3$ at high T: limiting Stefan-Boltzmann type behavior

V.V., A. Pásztor, Z. Fodor, S. Katz, H. Stoecker, 1708.02852; S. Borsányi, QM2017

Summary

- Non-Gaussian fluctuations measures are suitable probes in the search of critical point in QCD
- Non-Gaussian fluctuations from Lattice QCD are good probes for hadronic interactions
- van der Waals interactions between baryons in HRG change qualitative behavior of fluctuations of conserved charges in the crossover region
- LQCD data at imaginary μ_B suggests presence of repulsive baryonic interactions with 2nd virial coefficient $b \sim 1$ fm^3 in the crossover region
- Nuclear liquid-gas transition manifests itself into non-trivial net-baryon fluctuations in regions of phase diagram probed by heavy-ion collisions

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Thanks for your attention!

Backup slides

Quantum statistical van der Waals fluid

Free energy of classical vdW fluid:

$$F(T,V,N) = F^{\rm id}(T,V-bN,N) - a\,\frac{N^2}{V}$$

Ansatz: $F^{\rm id}(T,V-bN,N)$ is free energy of ideal *quantum* gas

$$\text{Pressure:} \qquad p = -\left(\frac{\partial F}{\partial V}\right)_{T,N} = p^{\rm id}(T,\mu^*) - a\,n^2$$

$$\text{Particle density:} \qquad n = \left(\frac{\partial p}{\partial \mu}\right)_{T} = \frac{n^{\rm id}(T,\mu^*)}{1 + b\,n^{\rm id}(T,\mu^*)}$$

$$\text{Shifted chemical potential:} \qquad \mu^* = \mu - b\,p - a\,b\,n^2 + 2\,a\,n$$

Model properties:

- Reduces to classical vdW equation when quantum statistics are negligible
- Reduces to ideal quantum gas for $a = 0$ and $b = 0$
- Entropy density non-negative and $s \to 0$ with $T \to 0$

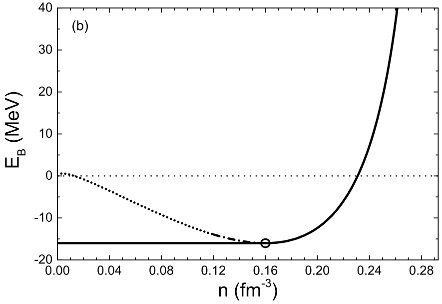
V.V., Anchishkin, Gorenstein, JPA '15 and PRC '15; Redlich, Zalewski, APPB '16.
a=0 $\Rightarrow$ excluded-volume model, D. Rischke et al., ZPC '91

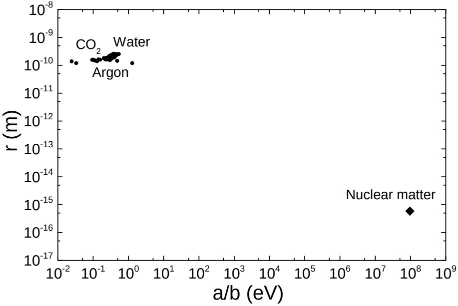
vdW gas of nucleons: zero temperature

How to fix a and b? For classical fluid usually tied to CP location.
Different approach: Reproduce saturation density and binding energy

From $E_B = E/A \cong -16$ MeV and $n = n_0 \cong 0.16$ fm^{-3} at $T = 0$ and $p = 0$

$$a \cong 329 \text{ MeV fm}^3 \text{ and } b \cong 3.42 \text{ fm}^3$$





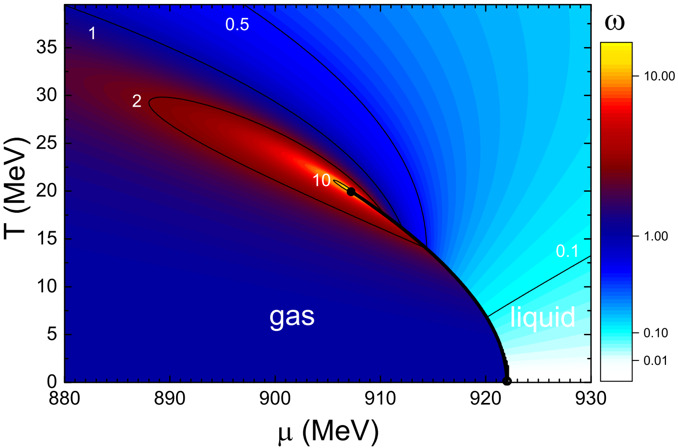
Mixed phase at $T = 0$ is specific:
A mix of vacuum ($n = 0$) and liquid at $n = n_0$

vdW eq. now at very different scale!

vdW gas of nucleons: scaled variance

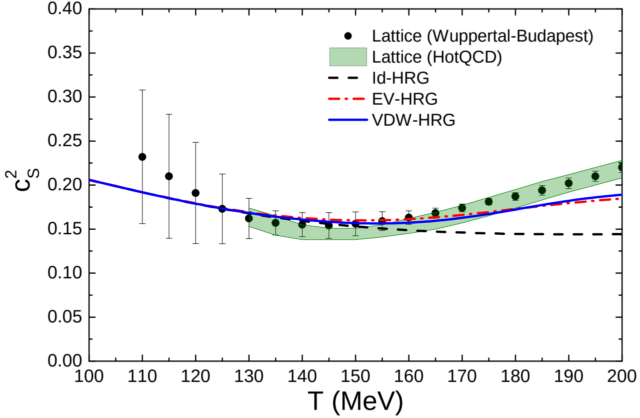
Scaled variance in quantum vdW:

$$\omega[N] = \omega_{\rm id}(T,\mu^*)\,\left[\frac{1}{(1-bn)^2} - \frac{2an}{T}\,\omega_{\rm id}(T,\mu^*)\right]^{-1}$$



vdW-HRG at $\mu_B = 0$: speed of sound

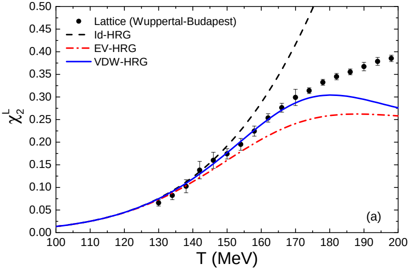
Speed of sound $c_s^2 = \frac{dp}{d\varepsilon}$



- Monotonic decrease in Id-HRG, at odds with lattice
- Minimum for EV-HRG/VDW-HRG at 150-160 MeV
- No acausal behavior, often an issue in models with eigenvolumes

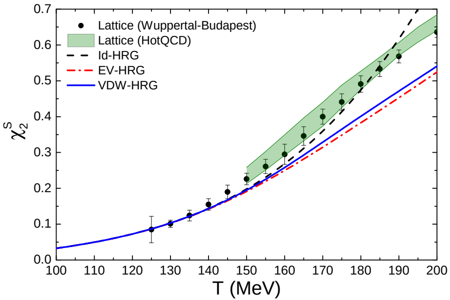
V.V., M. Gorenstein, H. Stoecker, Phys. Rev. Lett. 118, 182301 (2017)

vdW-HRG at $\mu = 0$: net-light and net-strangeness



- Net number of light quarks χ_2^L
- $L = (u + d)/2 = (3B + S)/2$
- Improved description in vdW-HRG

- Net-strangeness χ_2^S
- Underestimated by HRG models, similar for χ_{11}^{BS}
- Extra strange states?[1]
- Weaker vdW interactions for strange baryons?[2]

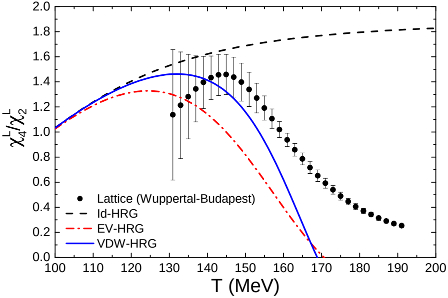


[1] Bazavov et al., PRL 113, 072001 (2014)
[2] Alba, Vovchenko, Gorenstein, Stoecker, arXiv:1606.06542

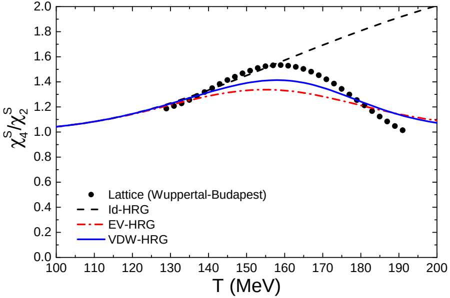
vdW-HRG at $\mu_B = 0$: net-light and net-strangeness

Fluctuations of net-light $L = (u+d)/2 = (3B+S)/2$ and net-strangeness

χ_4^L/χ_2^L



χ_4^S/χ_2^S



- Lattice shows peaked structures in crossover regions
- Not at all reproduced by ideal HRG, signal for deconfinement?[1]
- Peaks at different T for net-L and net-S $\Rightarrow$ flavor hierarchy?[2]
- vdW-HRG also shows peaks and flavor hierarchy $\Rightarrow$ cannot be traced back directly to deconfinement

[1] S. Ejiri, F. Karsch, K. Redlich, PLB 633, 275 (2006)
[2] Bellwied et al., PRL 111, 202302 (2013)