Towards the equation of state of hot QCD at finite baryon density

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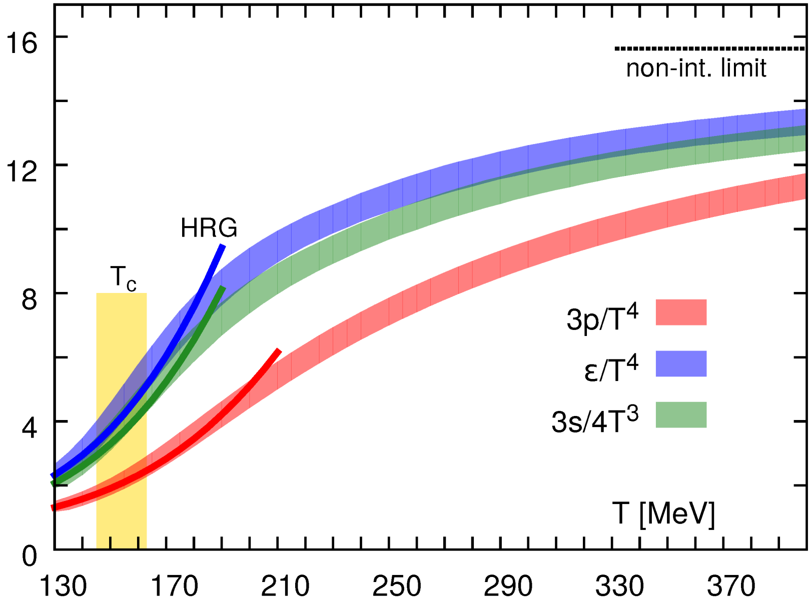
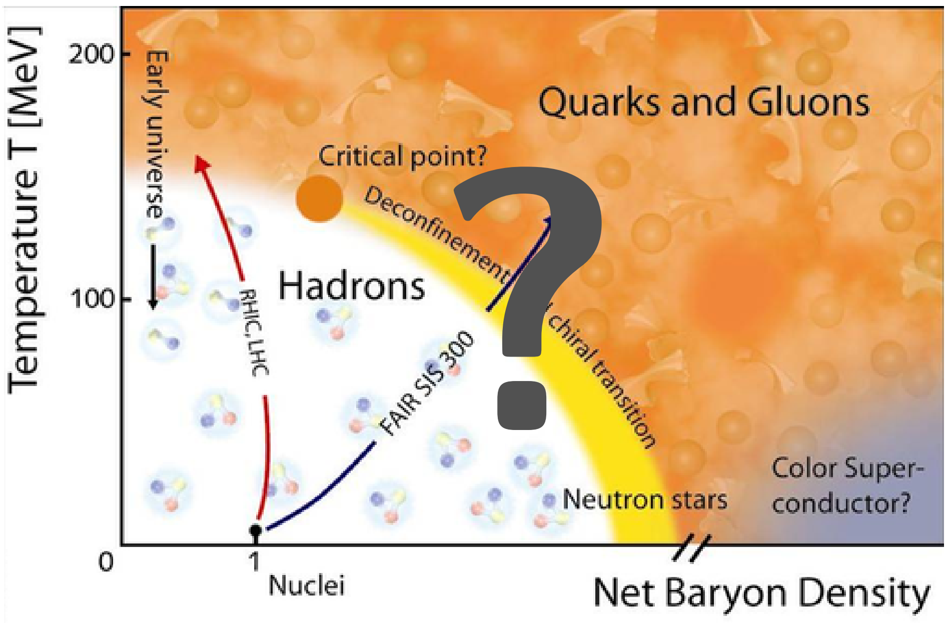
QCD phase diagram: towards finite density

?

$\mu_B = 0$ → $T - \mu_B$ plane





- QCD EoS at $\mu_B = 0$ available from first-principle lattice QCD simulations
- QCD EoS at finite density necessary for many applications, including hydro modeling of heavy-ion collisions at RHIC, SPS, FAIR energies
- Implementation of the QCD critical point necessary to look for its signatures

Non-zero μ_B and lattice QCD

At $\mu_B \neq 0$ fermion determinant is **complex**: $\det M[U,\mu] = |\det M[U,\mu]|\, e^{i\theta}$
"Probability distribution" interpretation is lost → lattice method **inapplicable**

Indirect lattice methods:

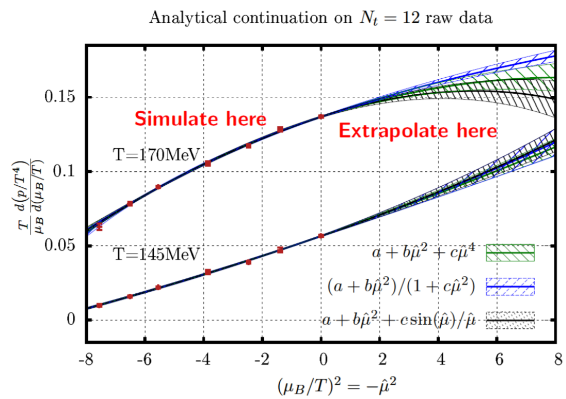
- Taylor expansion

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \frac{\chi_2^B(T,0)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T,0)}{4!}(\mu_B/T)^4 + \ldots$$

χ_k^B – cumulants (susceptibilities) of net baryon distribution
Can be computed in Lattice QCD at $\mu_B = 0$

- Analytic continuation from imaginary μ_B

No sign problem at $\mu_B = i\tilde{\mu}_B$.
Compute at $\mu_B^2 < 0$ and continue to $\mu_B^2 > 0$



[Wuppertal-Budapest collaboration, 1607.02493]

All these lattice methods inherently limited to "small" μ_B

A more **practical approach:** *use lattice data to constrain effective models*

Outline

1. Taylor expansion from lattice QCD
    - Model-independent method with a limited scope (small μ_B/T)
    - State-of-the-art and estimates for radius of convergence

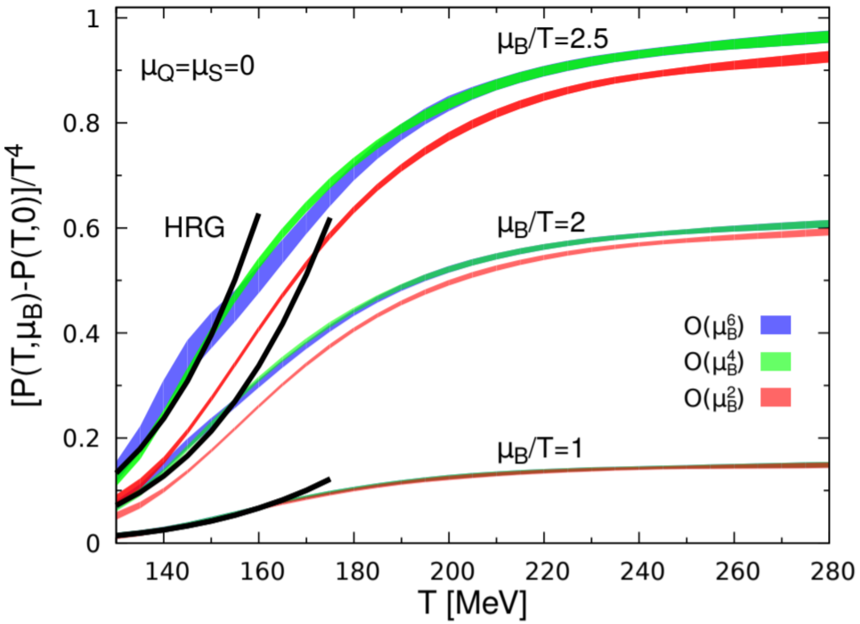
2. Lattice-based effective models
    - Cluster expansion model (CEM)
    - Hagedorn bag-like model
    - Chiral mean-field model

3. Status of the critical point at finite density

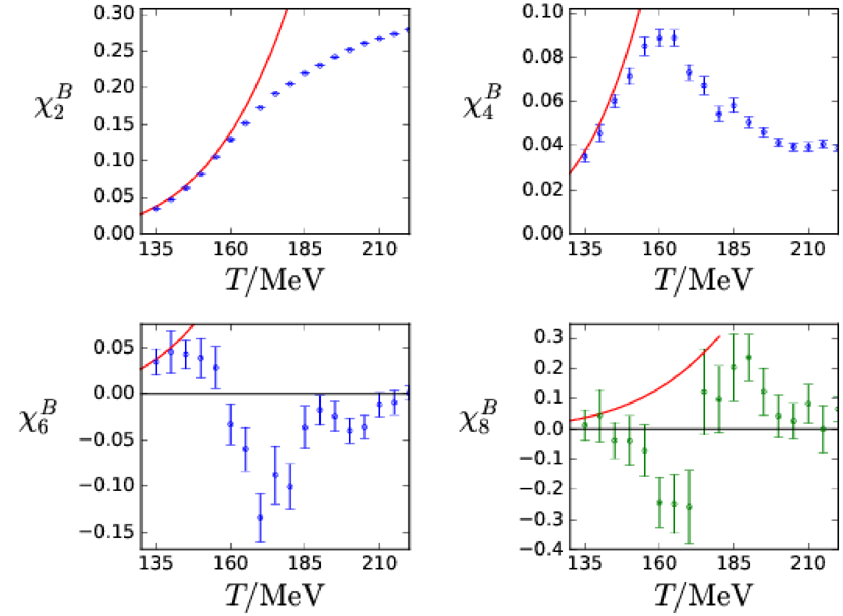
Finite μ_B EoS from Taylor expansion

$$\frac{p(T,\mu_B)}{T^4} = \frac{p(T,0)}{T^4} + \frac{\chi_2^B(T,0)}{2!}(\mu_B/T)^2 + \frac{\chi_4^B(T,0)}{4!}(\mu_B/T)^4 + \ldots$$

χ_k^B – cumulants of net baryon distribution, computed up to χ_8^B



[HotQCD collaboration, 1701.04325]



[Wuppertal-Budapest collaboration, 1805.04445]

- Off-diagonal susceptibilities also available → incorporate conservation laws $n_S = 0,\ n_Q/n_B = 0.4$
- Method inherently limited to "small" μ_B/T, within convergence radius

Taylor expansion and radius of convergence

A truncated Taylor expansion only useful within the radius of convergence. Its value is a priori unknown. Any singularity in *complex* μ_B plane will limit the convergence, it does not have to be a phase transition or a critical point

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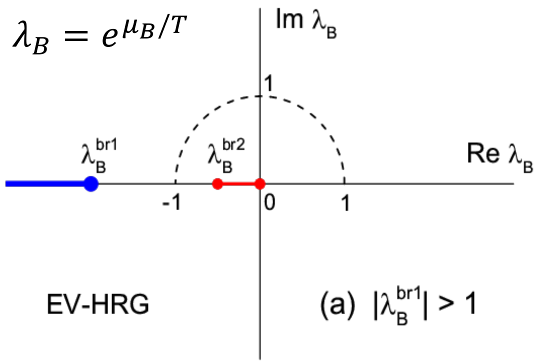
An example: HRG model with a baryonic excluded volume (EV)

$$V \to V - bN$$



$$p(T, \mu_B) \sim W\left[b\,\phi_B(T)\,e^{\mu_B/T}\right]$$

$b \simeq 1$ fm^3

Constrained to LQCD data
[**V.V.** et al., 1708.02852]

Lambert W(z) function has a branch cut singularity at $z = -e^{-1}$, corresponds to a negative (unphysical) fugacity

[Taradiy, Motornenko, **V.V.**, Gorenstein, Stoecker, 1904.08259]

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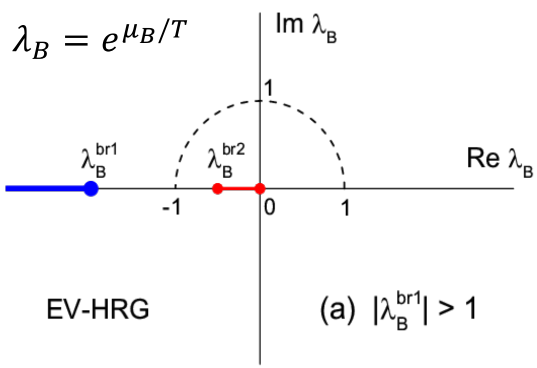

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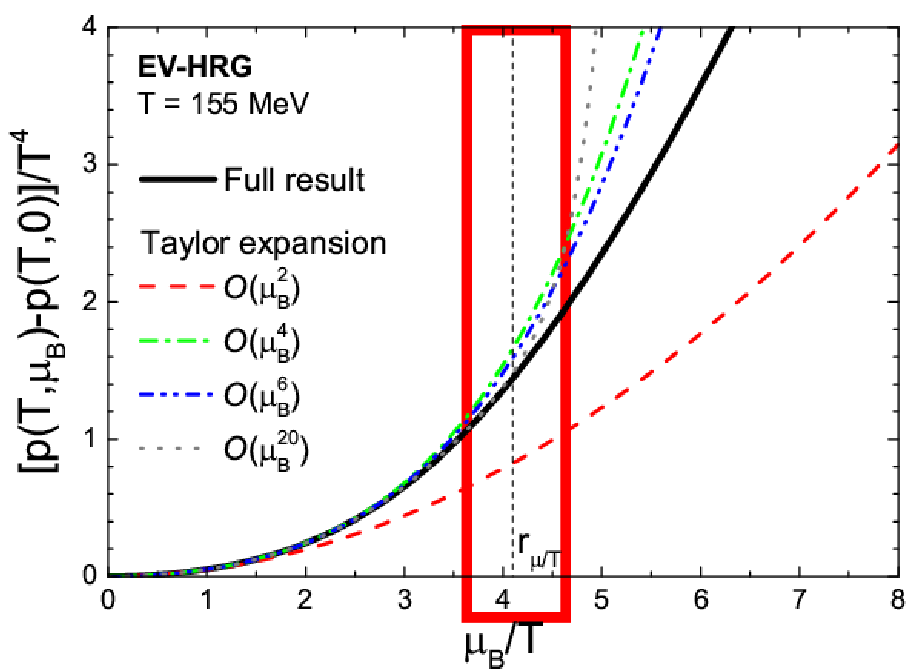


The best one can do with Taylor expansion

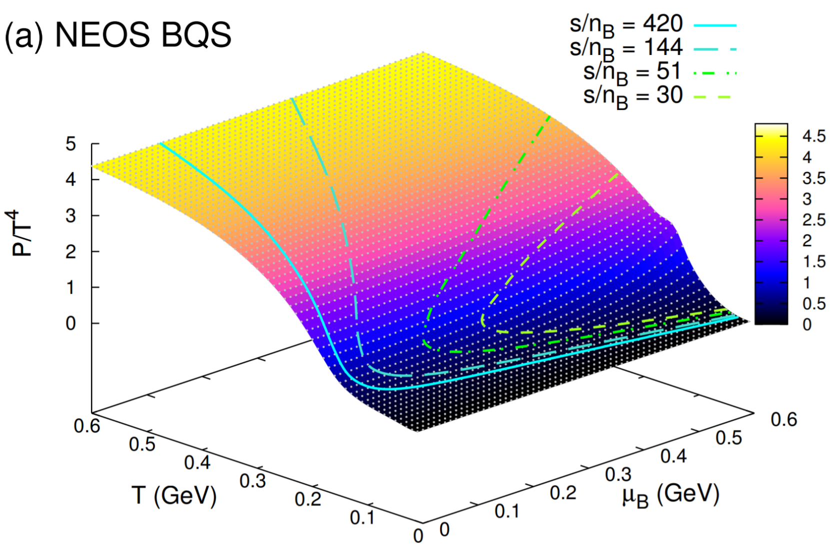
Truncated LQCD Taylor expansion

$$\frac{p}{T^4} = \sum_{i,j,k} \frac{\chi_{i,j,k}^{BQS}(T)}{i!\,j!\,k!} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$

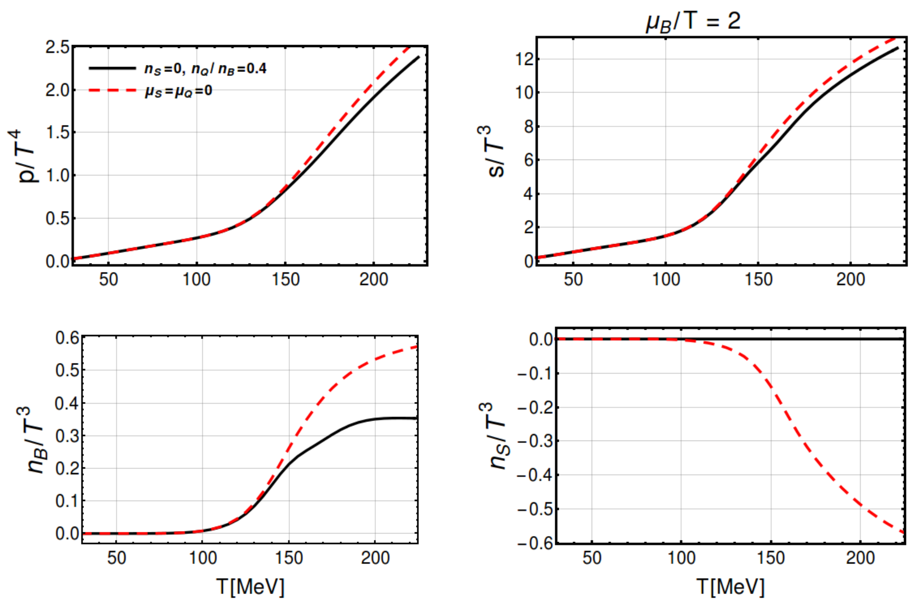
+

HRG model at smaller temperatures

$$\frac{p}{T^4} = \sum_{i \in \text{hrg}} T\,\phi_i^{\text{id}}(T)\, e^{b_i \mu_B/T}\, e^{q_i \mu_Q/T}\, e^{s_i \mu_S/T}$$



[Monnai, Schenke, Shen, 1902.05095]



[Noronha-Hostler, Parotto, Ratti, Stafford, 1902.06723]

- Includes the three conserved charges and conservation laws, no criticality
- Probably best one can do with Taylor expansion. Applications: RHIC BES

Truncated Taylor expansion and imaginary μ_B

Are we using all information available from lattice? Consider relativistic virial expansion (Laurent series in fugacity $p = \sum_{k=-\infty}^{\infty} p_{|k|}\, e^{k\mu_B/T}$) and imaginary μ_B

$$\left.\frac{\rho_B}{T^3}\right|_{\mu_B = i\theta_B T} = i \sum_{k=1}^{\infty} b_k(T) \sin(k\theta_B) \qquad \Rightarrow \qquad b_k(T) = -\frac{2i}{\pi} \int_0^{\pi} \frac{\rho_B(T, i\theta_B T)}{T^3} \sin(k\,\theta_B)\, d\theta_B$$

Relativistic virial/cluster expansion *Fourier coefficients*

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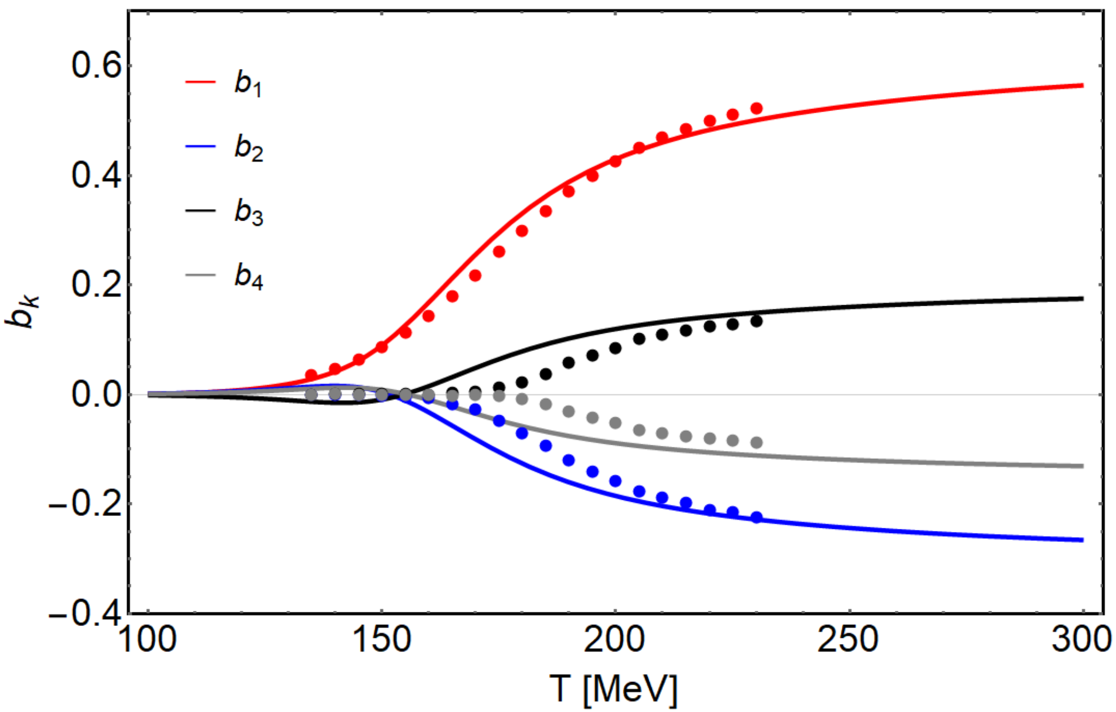
Relativistic virial/cluster expansion *Fourier coefficients*



Lines: *Taylor expansion up to χ_B^4 using lattice data, as in 1902.06723*

Symbols: *Lattice data for b_k from imaginary μ_B*

[**V.V.,** Pasztor, Fodor, Katz, Stoecker, 1708.02852]

Quite some room for improvement at T<200 MeV

Cluster Expansion Model — CEM

a model for QCD equation of state at finite baryon density
constrained to both susceptibilities and Fourier coefficients

V.V., *J. Steinheimer, O. Philipsen, H. Stoecker, Phys. Rev. D 97, 114030 (2018)*
V.V. *et al., Nucl. Phys. A 982, 859 (2019)*

Cluster Expansion Model (CEM)

Model formulation:

- Relativistic virial (cluster) expansion for baryon number density

$$\frac{\rho_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \sum_{k=1}^{\infty} b_k(T)\,\sinh(k\mu_B/T)$$

- $b_1(T)$ and $b_2(T)$ are model input from lattice QCD
- All higher order coefficients are predicted: $b_k(T) = \alpha_k^{SB}\,\frac{[b_2(T)]^{k-1}}{[b_1(T)]^{k-2}}$

Physical picture: Hadron gas with repulsion at moderate T, QGP-like phase at high T

Summed analytic form:

$$\frac{\rho_B(T,\mu_B)}{T^3} = -\frac{2}{27\pi^2}\,\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_1(x_+) - \mathrm{Li}_1(x_-)\right] + 3\left[\mathrm{Li}_3(x_+) - \mathrm{Li}_3(x_-)\right]\right\}$$

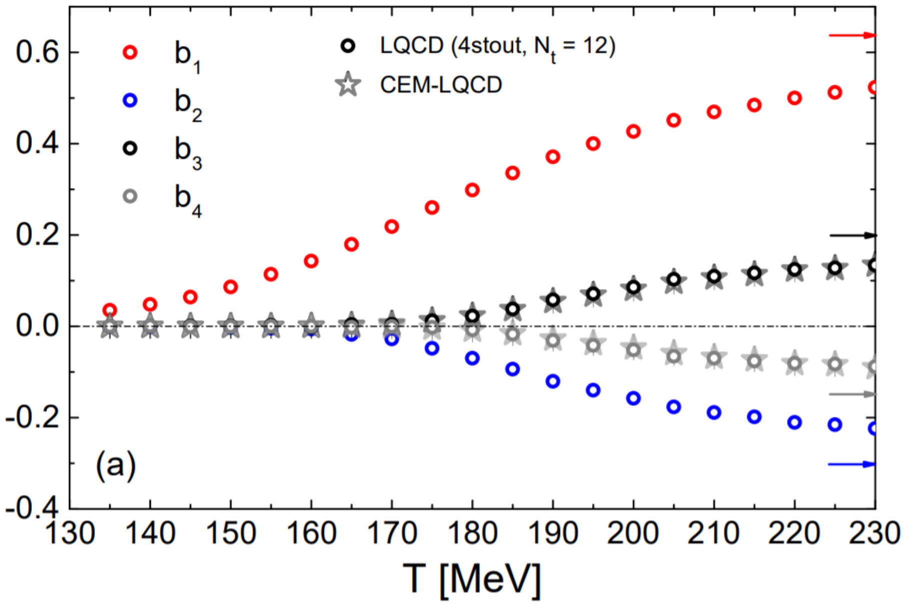
$$\hat{b}_k = \frac{b_k(T)}{b_k^{\mathrm{SB}}}, \quad x_\pm = -\frac{\hat{b}_2}{\hat{b}_1}\,e^{\pm\mu_B/T}, \quad \mathrm{Li}_s(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^s}$$

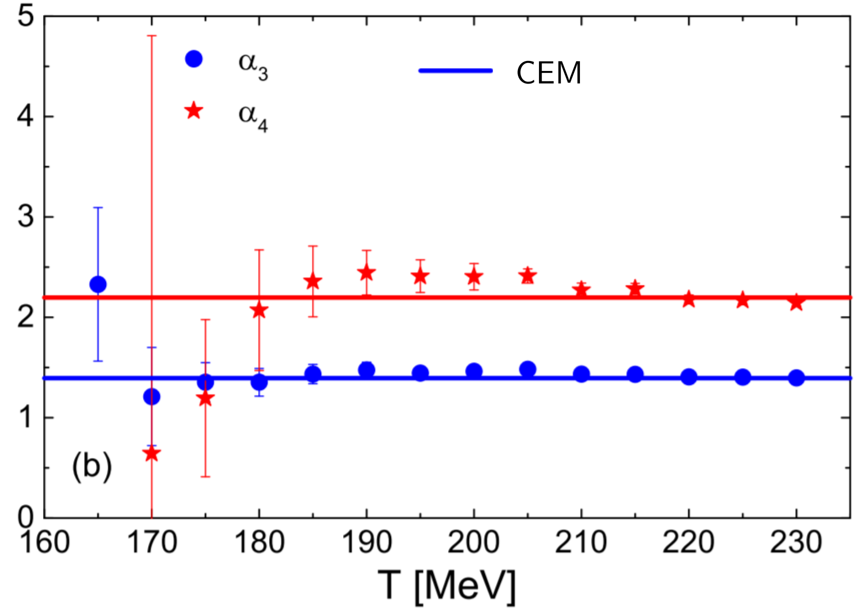
Regular behavior at real μ_B → *no-critical-point scenario*

CEM: Fourier coefficients

$b_k(T)$

$$\alpha_k(T) \equiv b_k(T)\,\frac{[b_1(T)]^{k-2}}{[b_2(T)]^{k-1}}$$



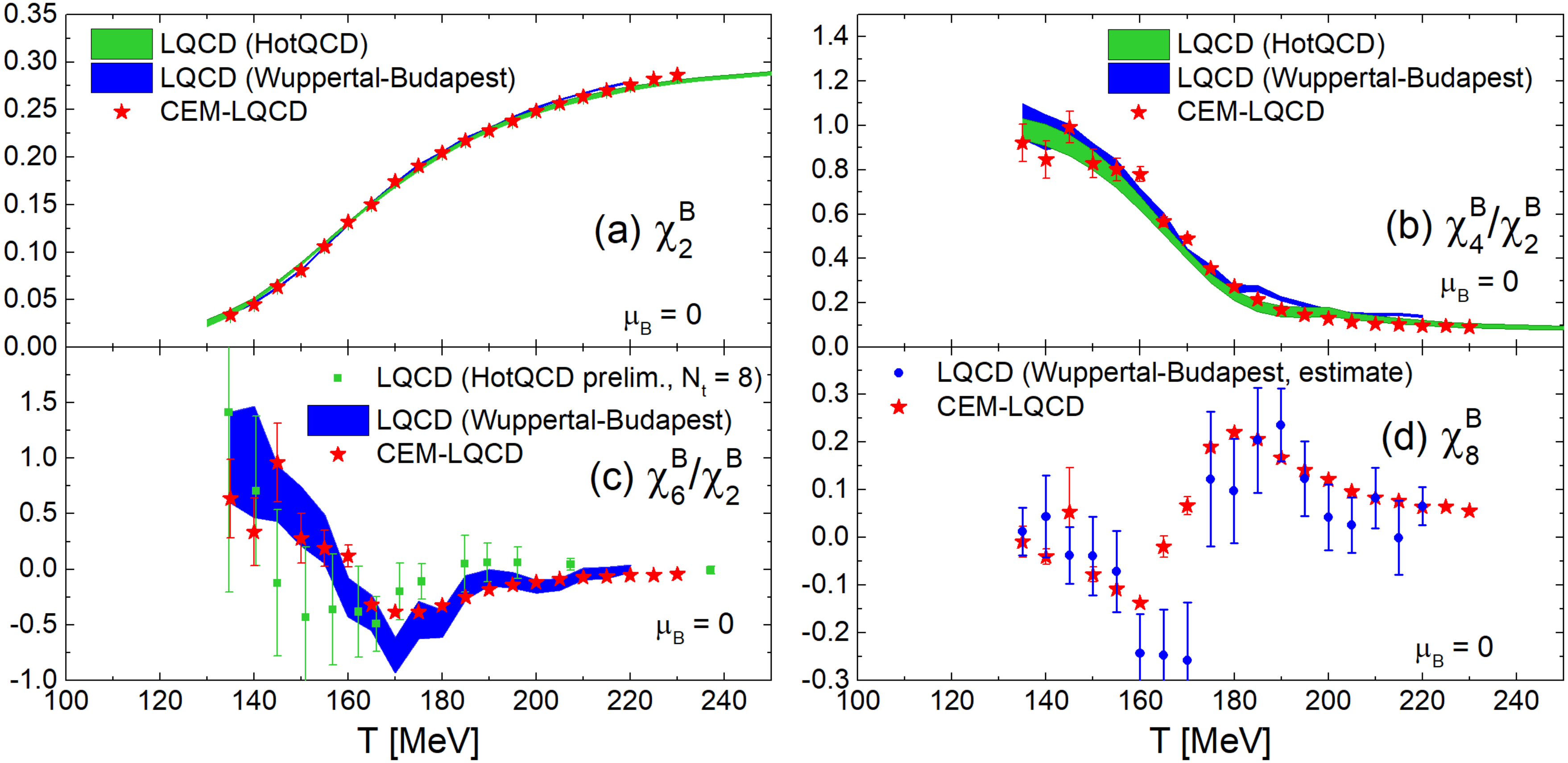


CEM: $b_1(T)$ and $b_2(T)$ as input → consistent description of $b_3(T)$ and $b_4(T)$

Lattice data on $b_{3,4}(T)$ inconclusive at $T \leq 170$ MeV

CEM: Baryon number susceptibilities

$$\chi_k^B(T,\mu_B) = -\frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_{2-k}(x_+) + (-1)^k\,\mathrm{Li}_{2-k}(x_-)\right] + 3\left[\mathrm{Li}_{4-k}(x_+) + (-1)^k\,\mathrm{Li}_{4-k}(x_-)\right]\right\}$$

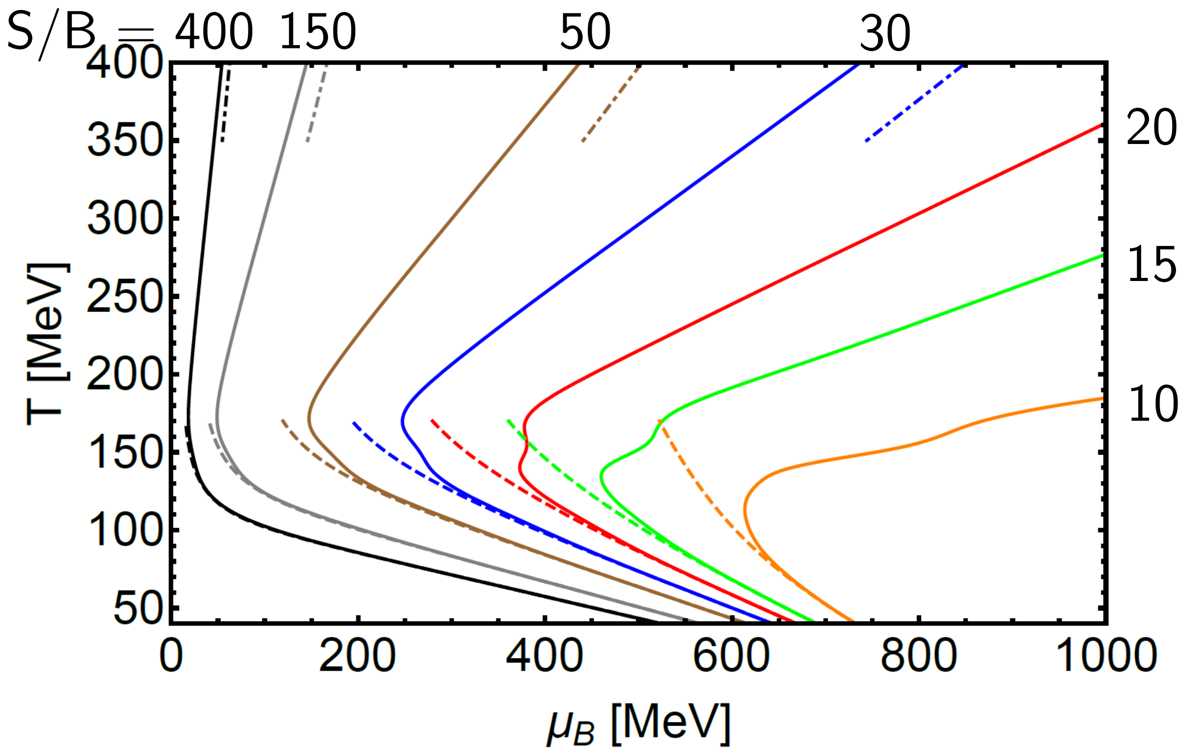


Lattice data from 1805.04445 (Wuppertal-Budapest), 1701.04325 & 1708.04897 (HotQCD)

CEM: Equation of state

$$\frac{p(T,\mu_B)}{T^4} = \frac{p_0(T)}{2} - \frac{2}{27\pi^2}\frac{\hat{b}_1^2}{\hat{b}_2}\left\{4\pi^2\left[\mathrm{Li}_2(x_+) - \mathrm{Li}_2(x_-)\right] + 3\left[\mathrm{Li}_4(x_+) - \mathrm{Li}_4(x_-)\right]\right\}$$

Input: $p_0(T),\ b_{1,2}(T)$ $\leftarrow$ parametrized LQCD + HRG



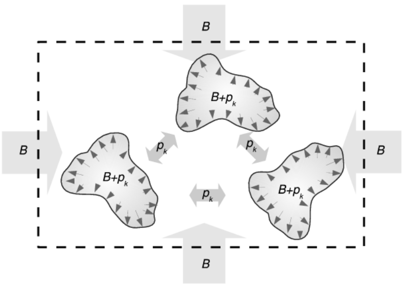
Tabulated CEM EoS available at https://fias.uni-frankfurt.de/~vovchenko/cem_table/

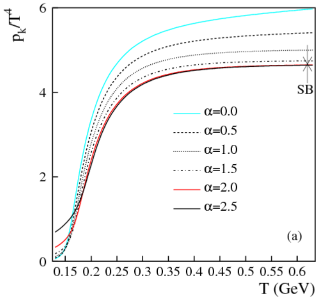
Currently restricted to single chemical potential (μ_B) and no critical point

Hagedorn (bag-like) resonance gas model with repulsive interactions

exactly solvable model with a (phase) transition between hadronic matter and QGP

[Gorenstein, Petrov, Zinovjev, PLB ’81; Gorenstein, W. Greiner, Yang, JPG ’98; Ferroni, Koch, PRC ‘09]



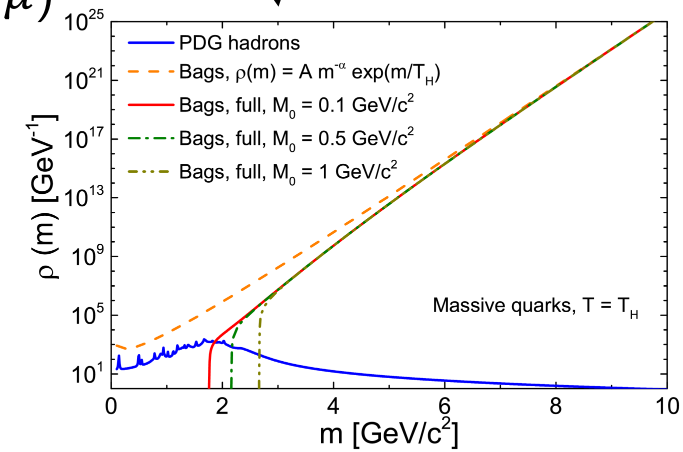


Here the model equation of state is constrained to lattice QCD

V.V., *M.I. Gorenstein, C. Greiner, H. Stoecker, Phys. Rev. C 99, 045204 (2019)*

Hagedorn bag-like model: formulation

- HRG + quark-gluon bags $\rho_Q(m,v) = C\, v^{\gamma}\, (m - Bv)^{\delta} \exp\left\{ \frac{4}{3} [\sigma_Q]^{1/4}\, v^{1/4}\, (m - Bv)^{3/4} \right\}$
- Non-overlapping particles (excluded volume correction) $V \to V - bN$
- Isobaric (pressure) ensemble $(T, V, \mu) \to (T, s, \mu)$
- *Massive* (thermal) partons (**new element**)



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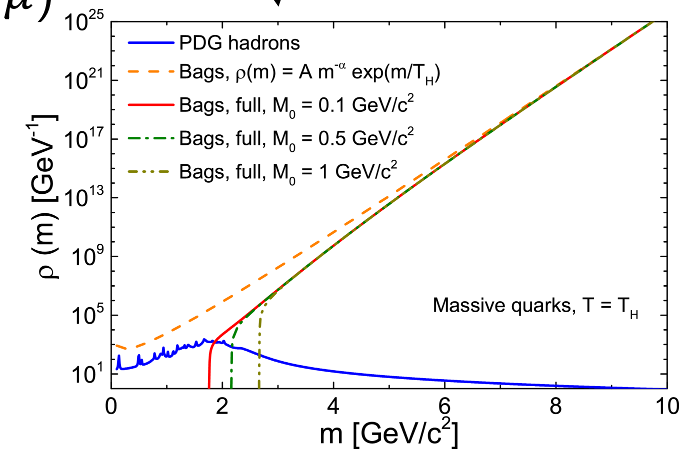
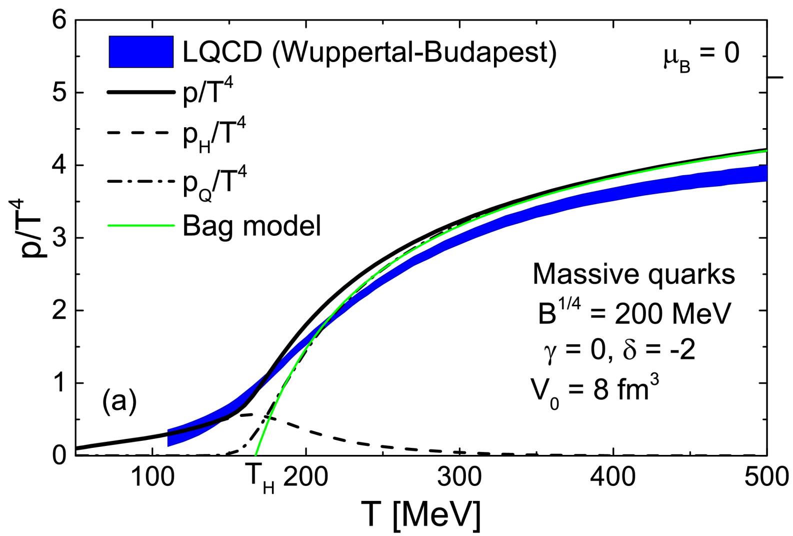
Resulting picture: transition (crossover, 1st order, 2nd order, etc.) between HRG and MIT bag model EoS, within single partition function



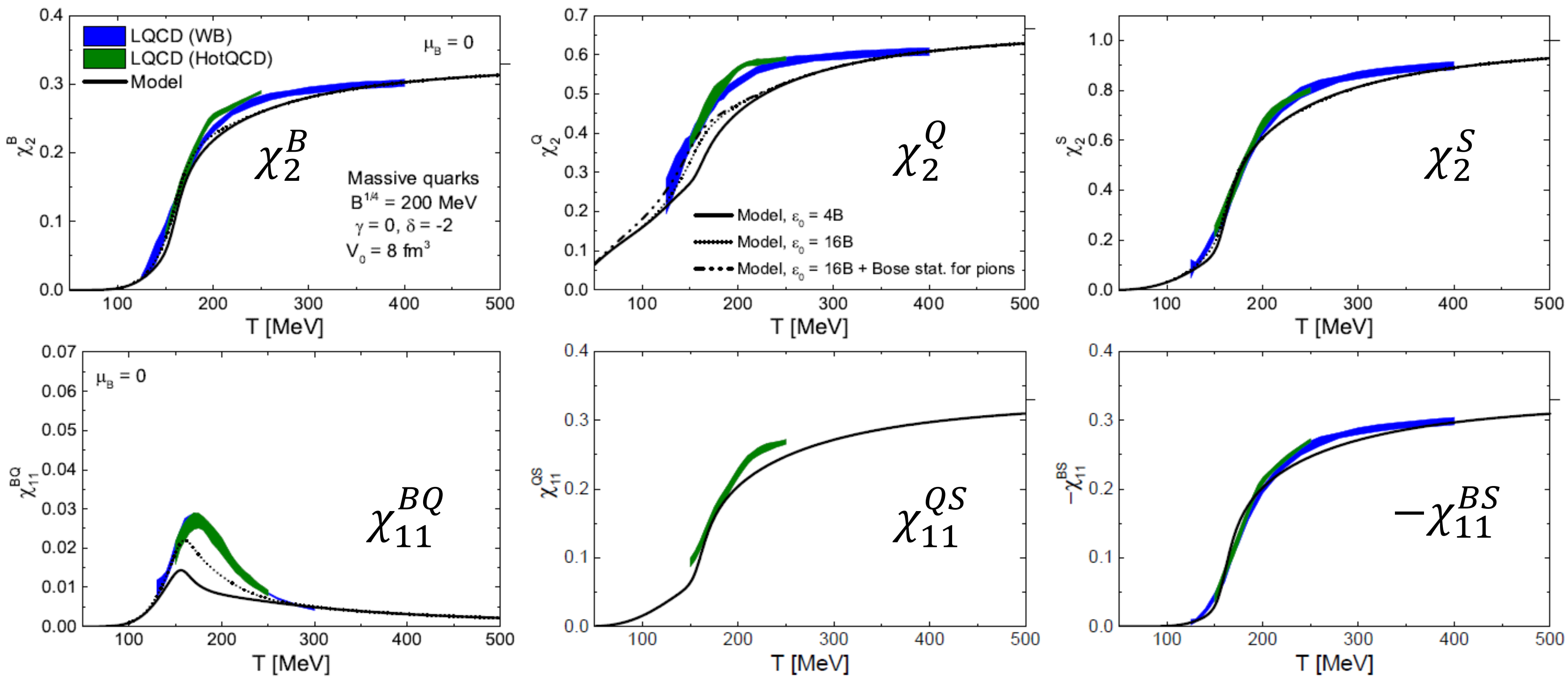


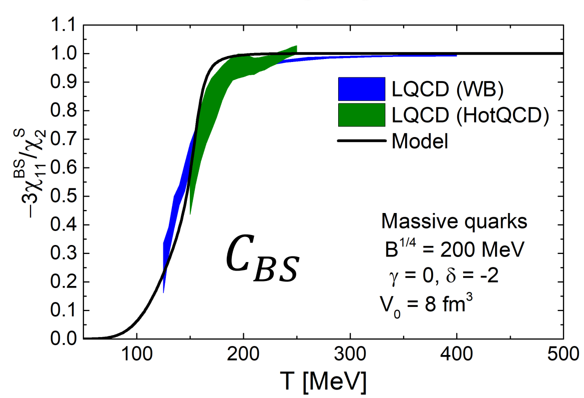
"Crossover" parameter set

$\gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8\text{ fm}^3$
$m_u = m_d = 300\text{ MeV}, \quad m_s = 350\text{ MeV}$
$m_g = 800\text{ MeV}, \quad B^{1/4} = 200\text{ MeV}$

$T_H \simeq 167\text{ MeV}$

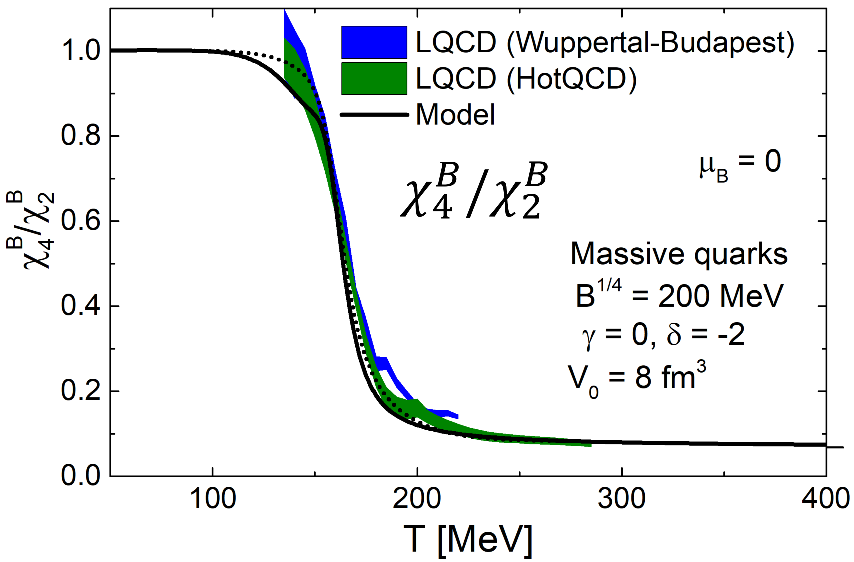
Hagedorn model: Susceptibilities

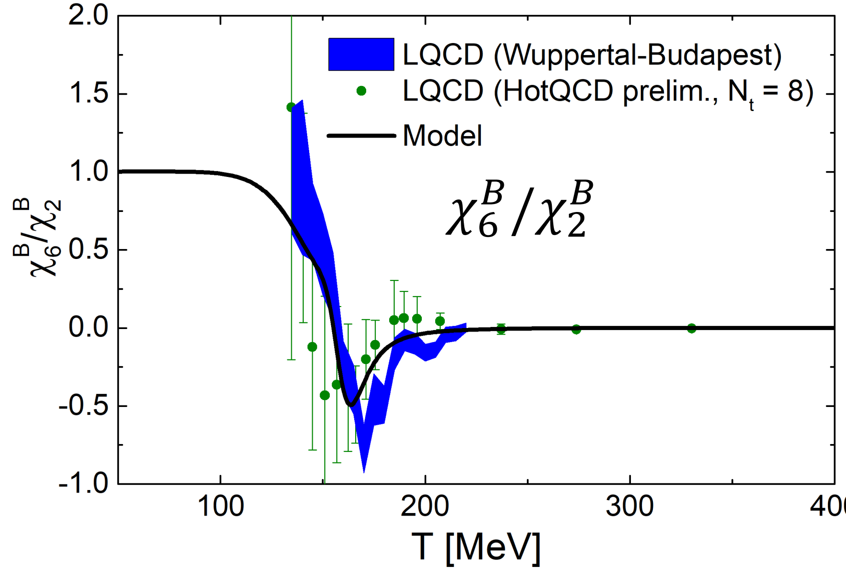


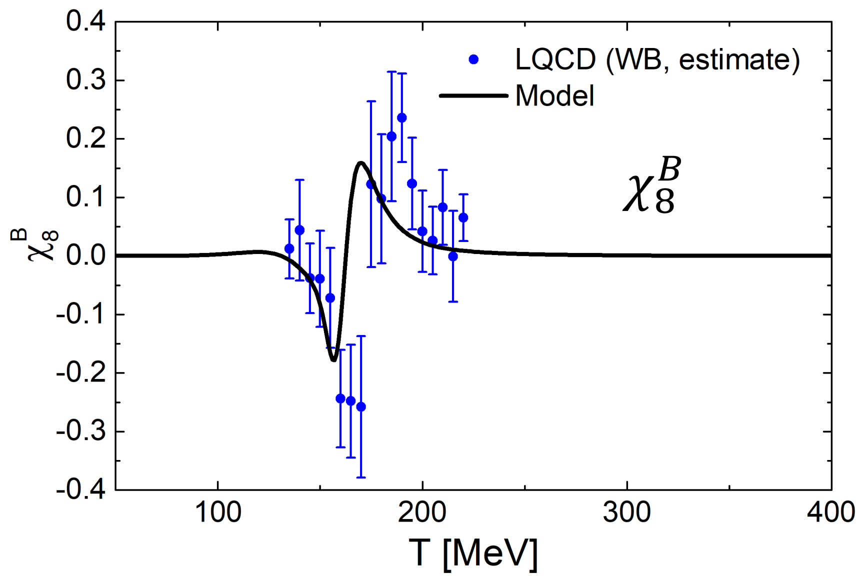


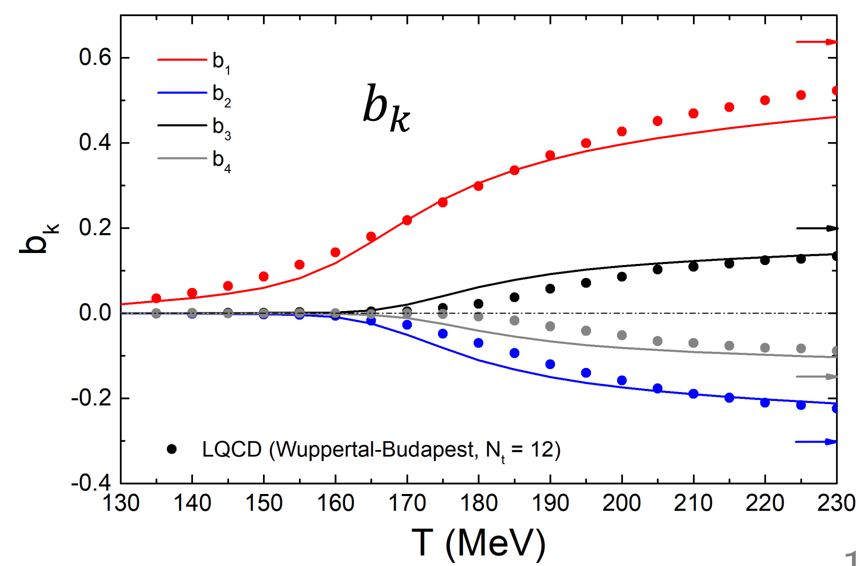
Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)

Hagedorn model: Susceptibilities and Fourier

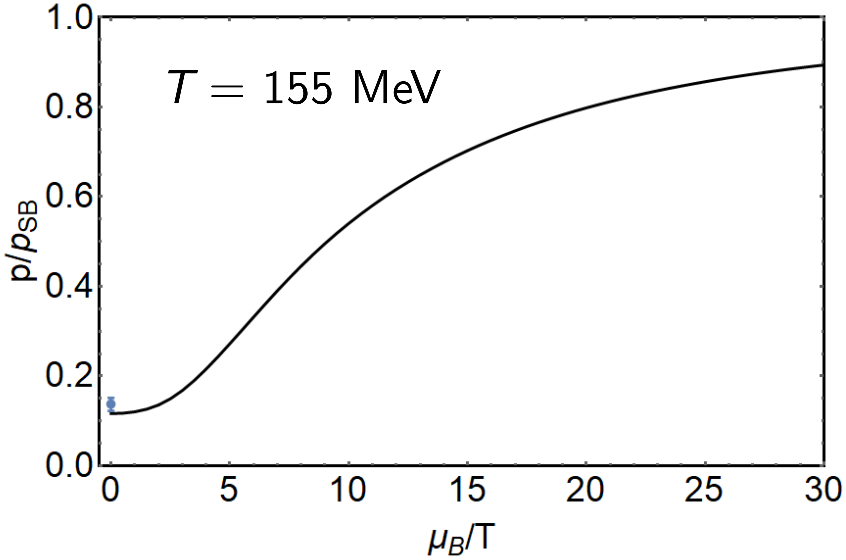


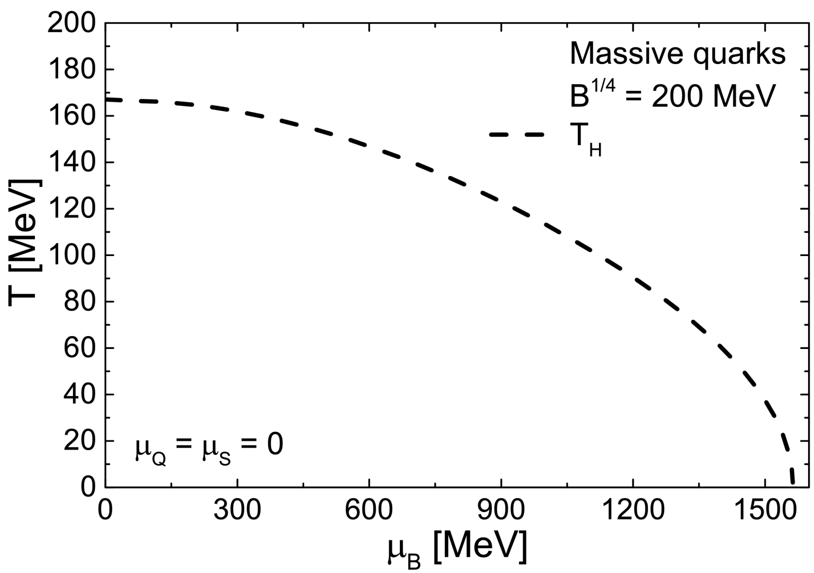






Hagedorn model: Finite baryon density





- Crossover transition to a QGP-like phase in both the T and μ_B directions
- Essentially a built-in "switching" function between HRG and QGP, thermodynamically consistent by construction (single partition function)
- Critical point/phase transition at finite μ_B can be incorporated through μ_B-dependence of γ and δ exponents in bag spectrum

see Gorenstein, Gazdzicki, Greiner, Phys. Rev. C (2005)

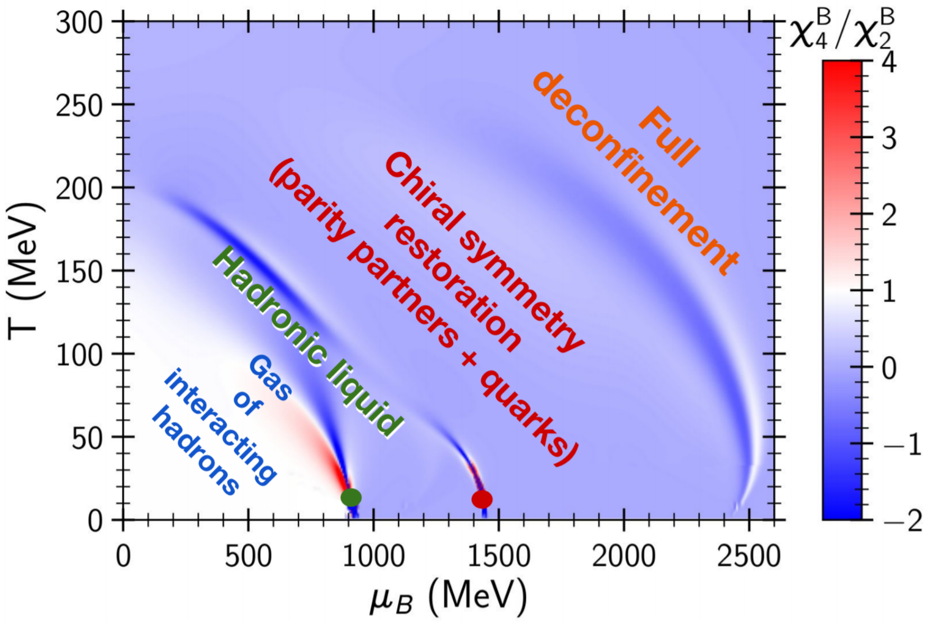
STRANGENESS IN QUARK MATTER SQM Bari 2019

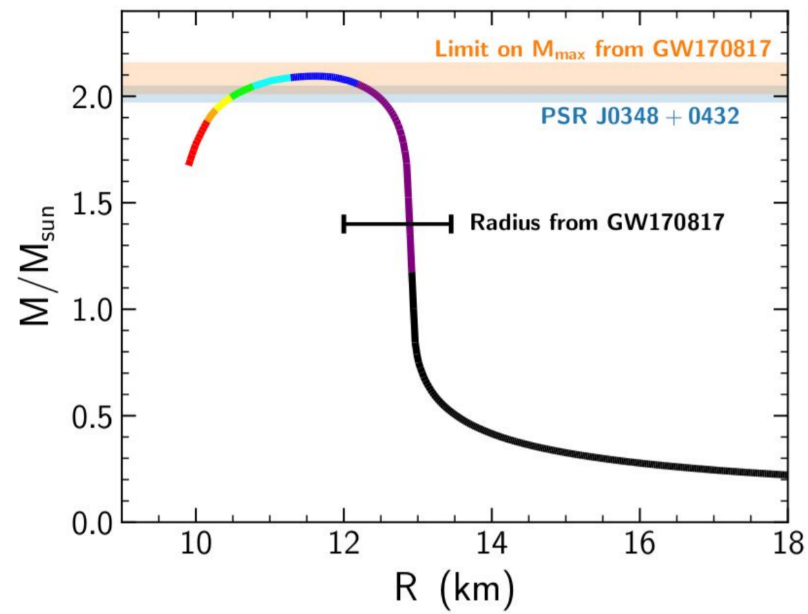
More on this at SQM2019

SU(3) parity-doublet quark-hadron chiral model

*Motornenko, **V.V.**, Steinheimer, Schramm, Stoecker, arxiv:1905.00866 & **A. Motornenko, talk this afternoon***

- Baryons interacting through mean fields + parity doubling + excluded volume
- Quarks in a PNJL-like approach
- Constrained to lattice data at $\mu_B = 0$ (high temperatures, low densities), empirical nuclear matter properties (low temperatures, high densities), neutron star properties, and gravitational-wave observations



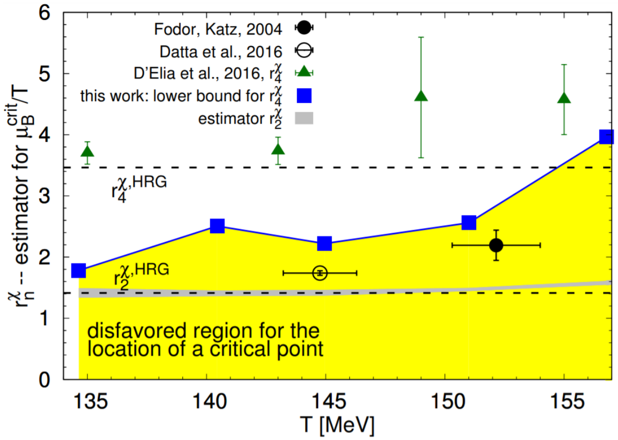


A considerably more involved approach needed for a "complete" phase diagram

Status of the critical point at finite density

Critical point: Lattice perspective

- Estimating radius of convergence of Taylor expansion from leading coeffs.



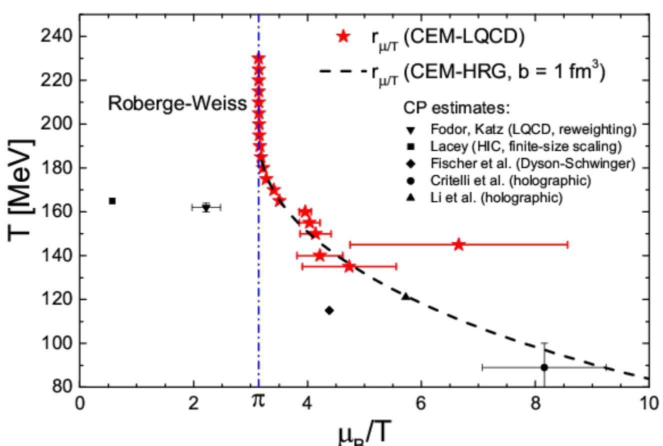
[HotQCD collaboration, 1701.04325]

No hints for a critical point at $T > 135$ MeV

"Small" $\mu_B/T < 2$-3 disfavored

[see also A. Pasztor (Wuppertal-Budapest), 1807.09862]

- Analysis of the relativistic virial (cluster) expansion



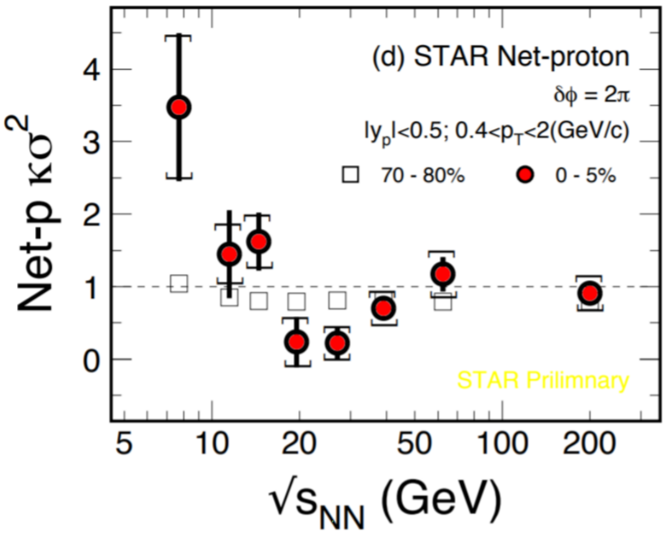
[**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]

Expansion coefficients consistent with a Roberge-Weiss like (Im $[\mu_B/T] = \pi$) transition in the complex plane

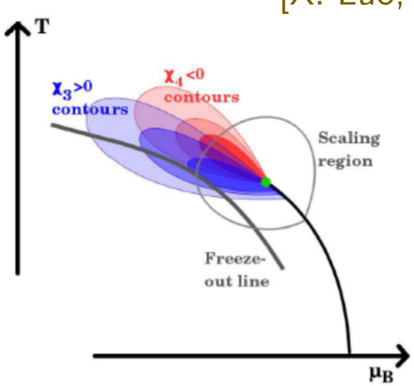
Critical point at $\mu_B/T < \pi$ disfavored

Critical point: Heavy-ion perspective

Measurements of (high-order) fluctuations and correlations



[X. Luo, CPOD2014]



[M. Stephanov, '09]

STAR measurement of net-proton kurtosis shows non-monotonic energy dependence which might be associated with criticality

Proper interpretation is challenging and requires dynamical modeling of critical fluctuations (critical mode) on top of hydro description + EoS with a critical point

Equations of state with a critical point:

3D-Ising + Taylor [P. Parotto et al., 1805.05249], switching function [C. Plumberg et al., 1812.01684], Hagedorn bag-like model [**V.V.** et al., in preparation], etc.

Implementing critical dynamics:

[Stephanov, Yin, 1712.10305; Nahrgang et al., 1804.05728; Akamatsu et al., 1811.05081]

All actively being developed

Summary

- Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate μ_B can be obtained in effective models constrained to all available lattice data, including *both* the Taylor expansion coefficients and Fourier coefficients of the cluster expansion

  Examples: Cluster Expansion Model, Hagedorn bag-like model, Chiral mean-field model etc.

- No critical point signals from lattice. "Small" $\mu_B/T < 2$-3 disfavored. Moderate collision energies are more promising in the search for the critical point.

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- Steady progress from lattice QCD on observables which constrain EoS at finite density. Reasonable (crossover) equation of state at moderate μ_B can be obtained in effective models constrained to all available lattice data, including *both* the Taylor expansion coefficients and Fourier coefficients of the cluster expansion

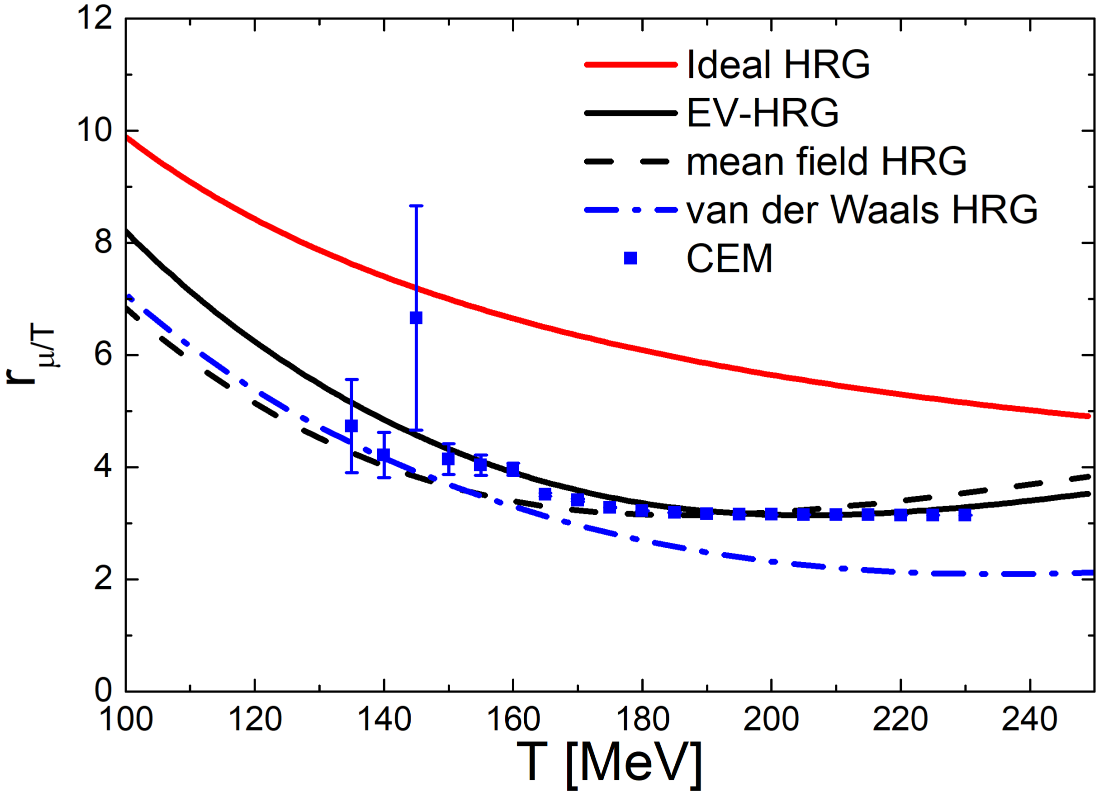
Examples: Cluster Expansion Model, Hagedorn bag-like model, Chiral mean-field model etc.

- No critical point signals from lattice. “Small” $\mu_B/T < 2$-3 disfavored. Moderate collision energies are more promising in the search for the critical point.

Thanks for your attention!

Backup slides

Radius of convergence from different models



Ideal HRG

Singularity in the nucleon Fermi-Dirac function

$$\left[\exp\left(\frac{\sqrt{m^2+p^2}-\mu_B}{T}\right)+1\right]^{-1}$$

EV-HRG & mean-field HRG

[**V.V.**+, 1708.02852] [Huovinen, Petreczky, 1708.02852]

Repulsive baryonic interactions. Singularity of the Lambert W function

van der Waals HRG

[**V.V.**, Gorenstein, Stoecker, 1609.03975]

Crossover singularities connected to the nuclear matter critical point at $T \sim 20$ MeV and $\mu_B \sim 900$ MeV

see also M. Stephanov, hep-lat/0603014

Cluster Expansion Model (CEM)

[**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]

Roberge-Weiss like transition: Im $\frac{\mu_B}{T} = \pi$

Taylor expansion likely divergent at $\mu_B/T \geq$ 3-5, regardless of existence of the QCD critical point

Cluster expansion in fugacities

Expand in fugacity $\lambda_B = e^{\mu_B/T}$ instead of μ_B/T – a relativistic analogue of Mayer's cluster expansion:

$$\frac{p(T,\mu_B)}{T^4} = \frac{1}{2}\sum_{k=-\infty}^{\infty} p_{|k|}(T)\, e^{k\mu_B/T} = \frac{p_0(T)}{2} + \sum_{k=1}^{\infty} p_k(T) \cosh(k\mu_B/T)$$

Net baryon density: $\frac{\rho_B(T,\mu_B)}{T^3} = \sum_{k=1}^{\infty} b_k(T) \sinh(k\mu_B/T), \quad b_k \equiv k p_k$

Analytic continuation to imaginary μ_B yields trigonometric Fourier series

$$\frac{\rho_B(T,\mathrm{i}\tilde{\mu}_B)}{T^3} = \mathrm{i}\sum_{k=1}^{\infty} b_k(T)\, \sin\left(\frac{k\,\tilde{\mu}_B}{T}\right)$$

with Fourier coefficients $b_k(T) = \frac{2}{\pi\, T^4} \int_0^{\pi T} d\tilde{\mu}_B \, [\mathrm{Im}\, \rho_B(T, i\tilde{\mu}_B)] \, \sin(k\,\tilde{\mu}_B/T)$

Four leading coefficients b_k computed in LQCD at the physical point

[**V.V.**, A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852]

Why cluster expansion is interesting?

Convergence properties of cluster expansion determined by singularities of thermodynamic potential in complex fugacity plane → encoded in the asymptotic behavior of the Fourier coefficients b_k

Examples:

- ideal quantum gas $$b_k \sim (\pm 1)^{k-1} \frac{e^{-km/T}}{k^{3/2}}$$ *Bose-Einstein condensation*

- cluster expansion model $$b_k \sim (-1)^{k-1} \frac{|\lambda_{\rm br}|^{-k}}{k}$$ *$|\lambda_{br}| = 1$ → Roberge-Weiss transition at imaginary μ_B*

  [**V.V.**, Steinheimer, Philipsen, Stoecker, 1711.01261]

- excluded volume model $$b_k \sim (-1)^{k-1} \frac{|\lambda_{\rm br}|^{-k}}{k^{1/2}}$$ *No phase transition, but a singularity at a negative λ*

  [Taradiy, **V.V.**, Gorenstein, Stoecker, in preparation]

- chiral crossover $$b_k \sim \frac{e^{-k\tilde{\mu}_c}}{k^{2-\alpha}} \sin(k\theta_c + \theta_0)$$ *Remnants of chiral criticality at $\mu_B = 0$*

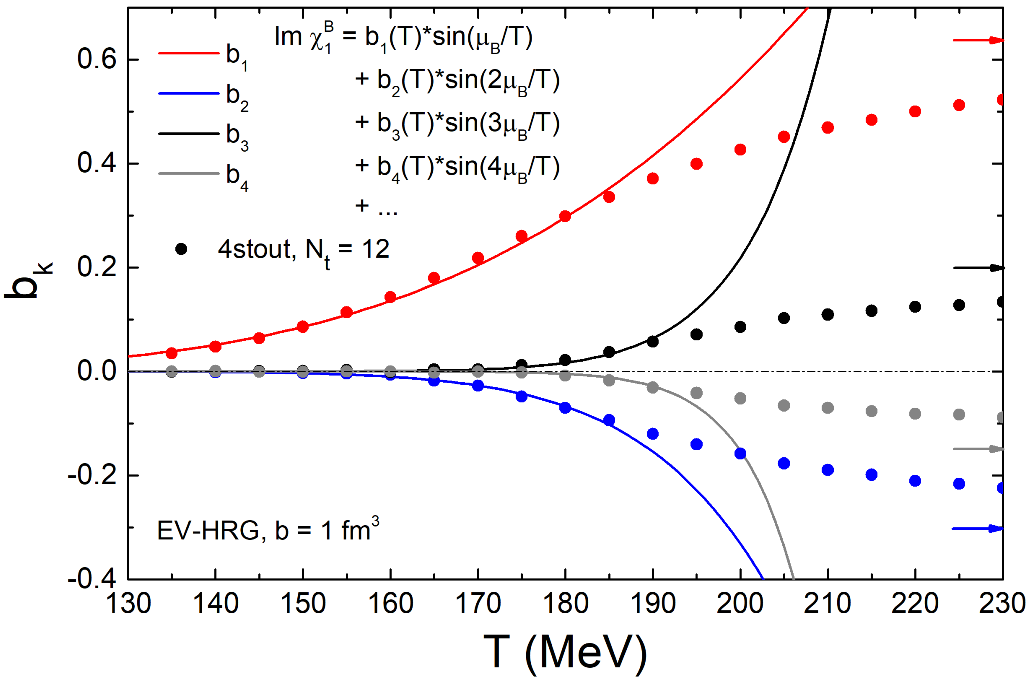
  [Almasi, Friman, Morita, Redlich, 1902.05457]

This work: signatures of a CP and a phase transition at finite density

HRG with repulsive baryonic interactions

Repulsive interactions with excluded volume (EV) $V \to V - bN$

[Hagedorn, Rafelski, '80; Dixit, Karsch, Satz, '81; Cleymans et al., '86; Rischke et al., Z. Phys. C '91]



HRG with baryonic EV:

$$p_B(T, \mu_B) = p_B^{\mathrm{id}}(T, \mu_B - b p_B)$$

$$b_k^{ev}(T) = (-1)^{k-1} \frac{2\,k^k}{k!} (b\,T^3)^{k-1} \left[\frac{\phi_B(T)}{T^3}\right]^k$$

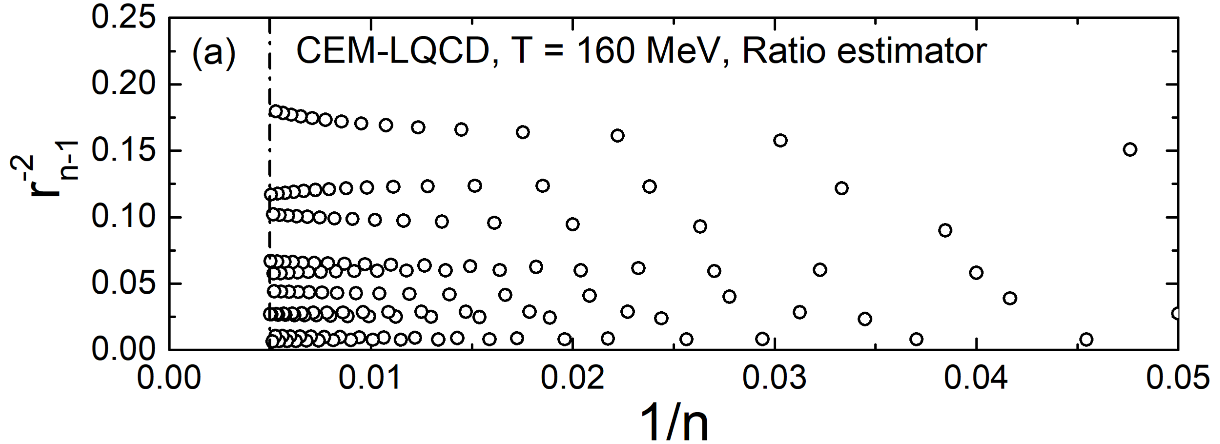
V.V., A. Pasztor, Z. Fodor, S.D. Katz, H. Stoecker, 1708.02852

- Non-zero $b_k(T)$ for $k \geq 2$ signal deviation from ideal HRG
- EV interactions between baryons ($b \approx 1$ fm^3) reproduce lattice trend

Using estimators for radius of convergence

a) Ratio estimator:

$$r_n = \left| \frac{(2n+2)(2n+1)\chi^B_{2n}}{\chi^B_{2n+2}} \right|^{1/2}$$

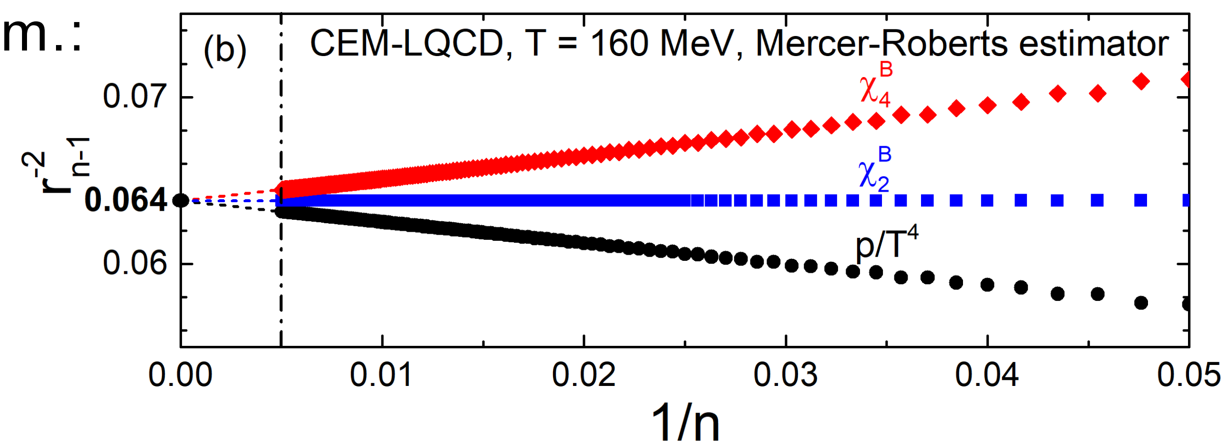


Ratio estimator is *unable* to determine the radius of convergence, nor to provide an upper or lower bound

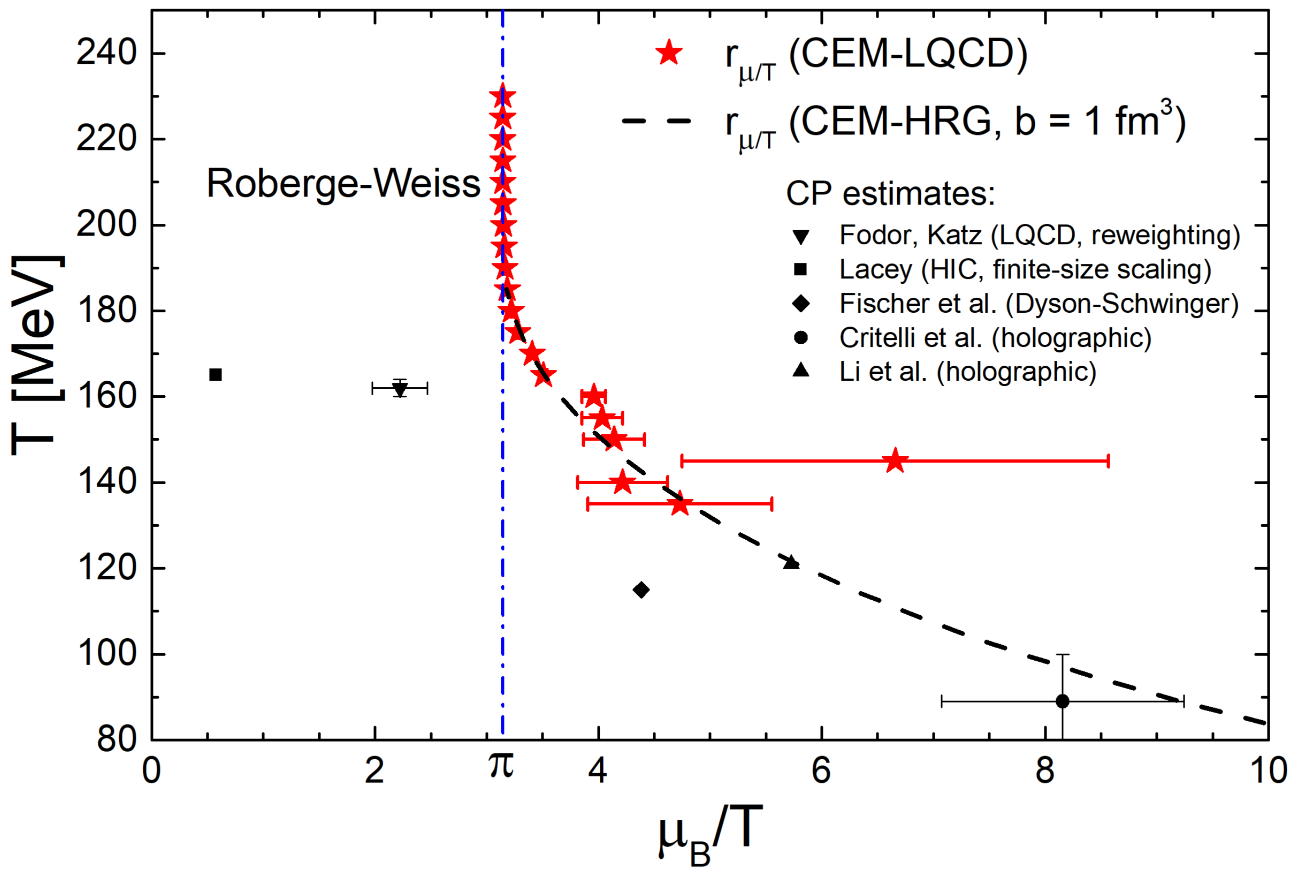
b) Mercer-Roberts estim.:

$$r_n = \left| \frac{c_{n+1}\, c_{n-1} - c_n^2}{c_{n+2}\, c_n - c_{n+1}^2} \right|^{1/4}$$

$$c_n = \frac{\chi^B_{2n}}{(2n)!}$$



CEM: Radius of convergence



Radius of convergence approaches Roberge-Weiss transition value

- At $T > T_{RW}$ expected $\left[\frac{\mu_B}{T}\right]_c = \pm i\pi$ [Roberge, Weiss, NPB '86] $T_{RW} \sim 208$ MeV [C. Bonati et al., 1602.01426]
- Complex plane singularities interfere with the search for CP

Expected asymptotics

- At low T/densities QCD $\simeq$ ideal hadron resonance gas

$$\frac{p^{\mathrm{hrg}}(T,\mu_B)}{T^4} = \frac{\phi_M(T)}{T^3} + 2\,\frac{\phi_B(T)}{T^3}\cosh\left(\frac{\mu_B}{T}\right),$$

$$\phi_B(T) = \sum_{i\in B}\int dm\,\rho_i(m)\,\frac{d_i\,m^2\,T}{2\pi^2}\,K_2\left(\frac{m}{T}\right),$$

$$p_0^{hrg}(T) = \frac{\phi_M(T)}{T^3}, \quad p_1^{hrg}(T) = \frac{2\,\phi_B(T)}{T^3}, \quad p_k^{\mathrm{hrg}}(T) \equiv 0,\ k \geq 2$$

- At high T QCD $\simeq$ ideal gas of massless quarks and gluons

$$\frac{p^{\mathrm{SB}}(T,\mu_B)}{T^4} = \frac{8\pi^2}{45} + \sum_{f=u,d,s}\left[\frac{7\pi^2}{60} + \frac{1}{2}\left(\frac{\mu_B}{3T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_B}{3T}\right)^4\right],$$

$$p_0^{\mathrm{SB}} = \frac{64\pi^2}{135}, \quad p_k^{\mathrm{SB}} = \frac{(-1)^{k+1}}{k^2}\,\frac{4\,[3+4\,(\pi k)^2]}{27\,(\pi k)^2}, \quad b_k^{SB} = k\,p_k^{\mathrm{SB}}.$$

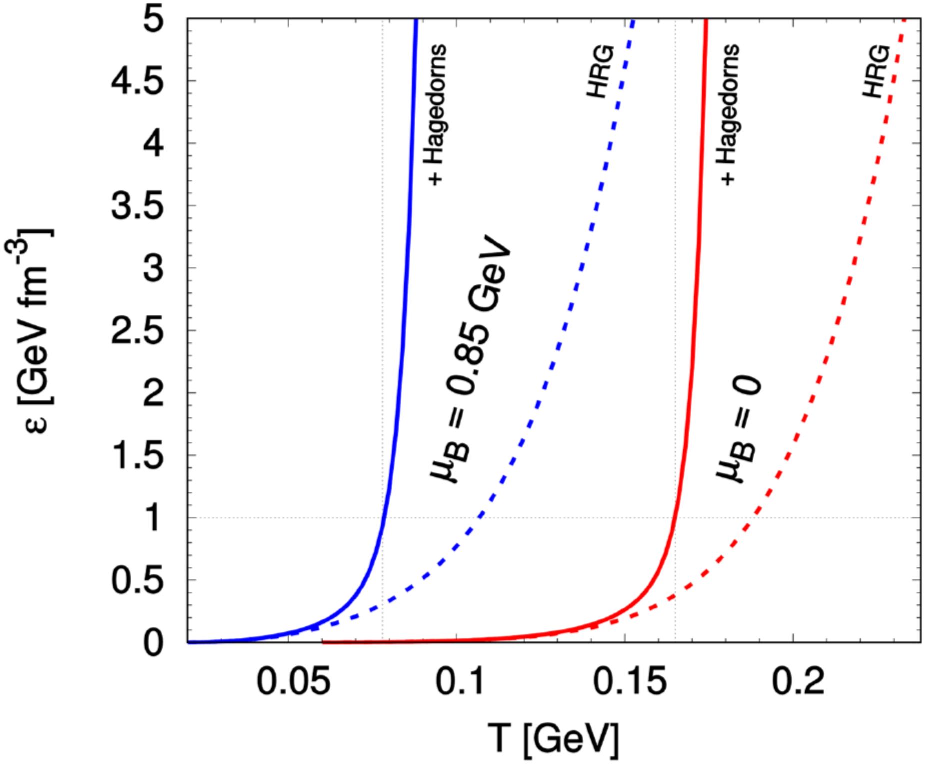
Lattice data explore intermediate, transition region $130 < T < 230$ *MeV*

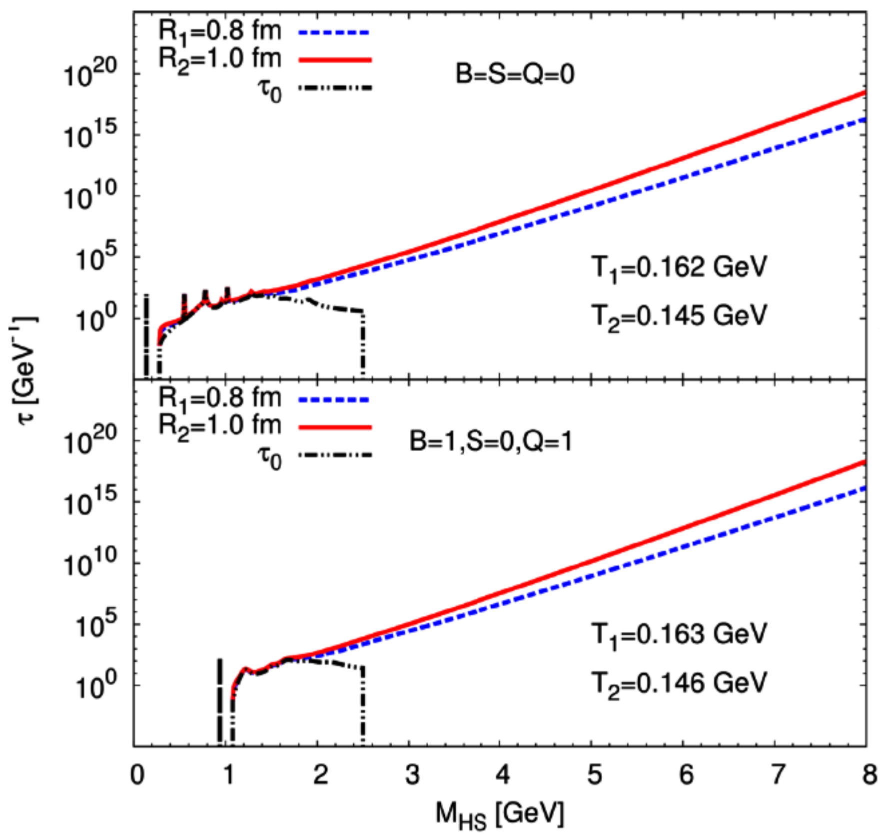
*In this study we assume that $\mu_S = \mu_Q = 0$

Hagedorn resonance gas

HRG + exponential Hagedorn mass spectrum, e.g. as obtained from the bootstrap equation [Hagedorn '65; Frautschi, '71]

$$\rho(m) = A\, m^{-\alpha} \exp(m/T_H)$$





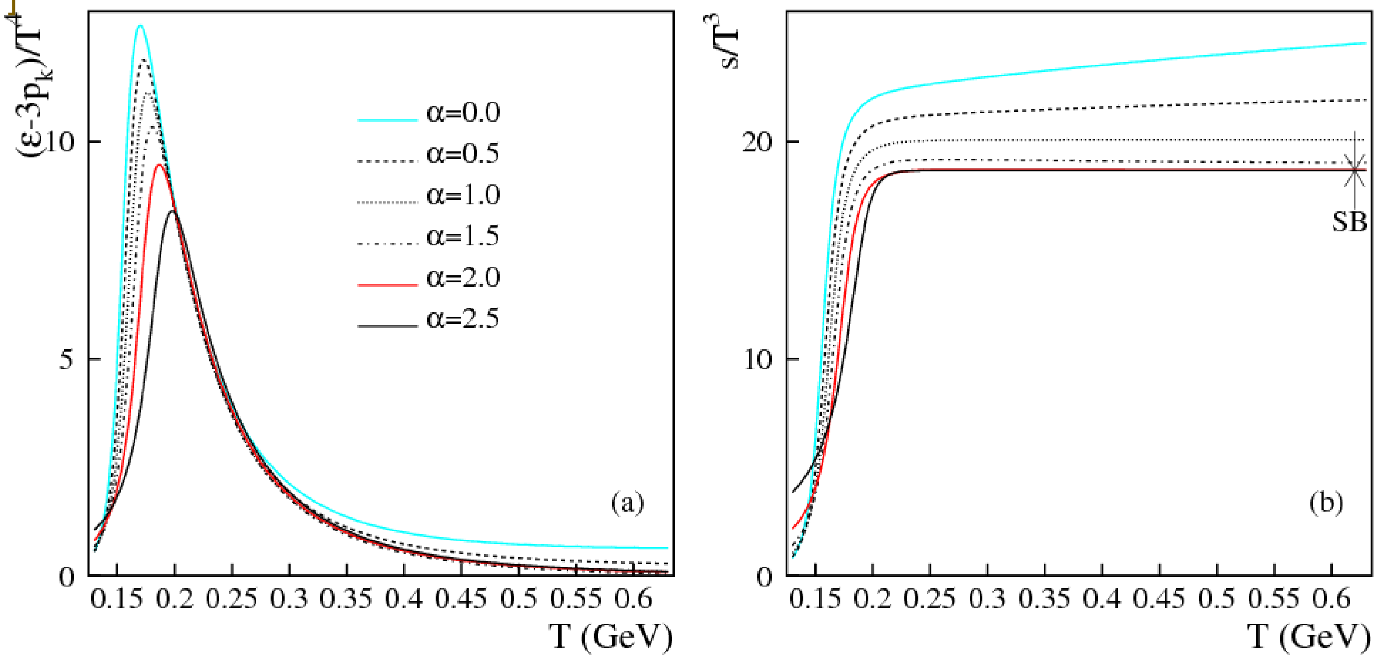
[Beitel, Gallmeister, Greiner, 1402.1458]

If Hagedorns are point-like, *T_H is the limiting temperature*

From limiting temperature to crossover

- A gas of extended objects → excluded volume
- Exponential spectrum of *compressible* QGP bags
- Both phases described by single partition function

[Gorenstein, Petrov, Zinovjev, PLB '81; Gorenstein, W. Greiner, Yang, JPG '98; I. Zakout et al., NPA '07]



[Ferroni, Koch, PRC 79, 034905 (2009)]

Crossover transition in bag-like model qualitatively compatible with LQCD

Model formulation

Thermodynamic system of known hadrons and quark-gluon bags

Mass-volume density $\rho(m, v; \lambda_B, \lambda_Q, \lambda_S) = \rho_H + \rho_Q$

$$\rho_H(m, v; \lambda_B, \lambda_Q, \lambda_S) = \sum_{i \in \mathrm{HRG}} \lambda_B^{b_i} \, \lambda_Q^{q_i} \, \lambda_S^{s_i} \, d_i \, \delta(m - m_i) \, \delta(v - v_i)$$

PDG hadrons

$$\rho_Q(m, v; \lambda_B, \lambda_Q, \lambda_S) = C \, v^{\gamma} \, (m - Bv)^{\delta} \, \exp\left\{ \frac{4}{3} [\sigma_Q v]^{1/4} \, (m - Bv)^{3/4} \right\} \theta(v - V_0) \, \theta(m - Bv)$$

Quark-gluon bags [J. Kapusta, PRC '81; Gorenstein+, ZPC '84]

Non-overlapping particles $\rightarrow$ **isobaric (pressure) ensemble**

[Gorenstein, Petrov, Zinovjev, PLB '81]

$$\hat{Z}(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int_0^{\infty} Z(T, V, \lambda_B, \lambda_Q, \lambda_S) \, e^{-sV} \, dV = [s - f(T, s, \lambda_B, \lambda_Q, \lambda_S)]^{-1}$$

$$f(T, s, \lambda_B, \lambda_Q, \lambda_S) = \int dv \int dm \, \rho(m, v; \lambda_B, \lambda_Q, \lambda_S) \, e^{-v\,s} \, \phi(T, m)$$

The system pressure is $p = T s^*$ with s^* being the *rightmost* singularity of $\hat{Z}$

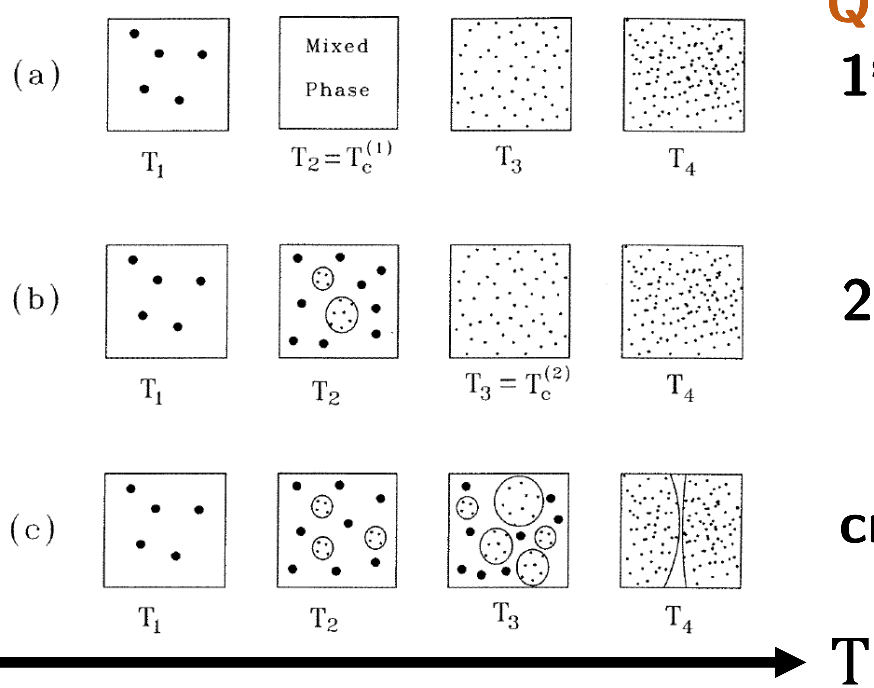
Mechanism for transition to QGP

The isobaric partition function, $\hat{Z}(T,s,\lambda) = [s - f(T,s,\lambda)]^{-1}$, has

- pole singularity $s_H = f(T, s_H, \lambda)$ **"hadronic" phase**
- singularity s_B in the function $f(T,s,\lambda)$ due to the exponential spec

$$p_B = T\, s_B = \frac{\sigma_Q}{3} T^4 - B$$

MIT bag model EoS for QGP [Chodos+, PRD '74; Baacke, APPB '77]



1st order PT

"collision" of singularities $s_H(T_C) = s_B(T_C)$

2nd order PT

crossover $s_H(T) > s_B(T)$ at all T

Crossover transition

Type of transition is determined by exponents γ and δ of bag spectrum

Crossover seen in lattice, realized in model for $\gamma + \delta \geq -3$ and $\delta \geq -7/4$

[Begun, Gorenstein, W. Greiner, JPG '09]

Transcendental equation for pre

$$p(T, \lambda_B, \lambda_Q, \lambda_S) = T \sum_{i \in \mathrm{HRG}} d_i \, \phi(T, m) \, \lambda_B^{b_i} \, \lambda_Q^{q_i} \, \lambda_S^{s_i} \, \exp\left(-\frac{m_i p}{4BT}\right)$$

$$+ \frac{C}{\pi} \, T^{5+4\delta} \, [\sigma_Q]^{\delta+1/2} \, [B + \sigma_Q T^4]^{3/2} \left(\frac{T}{p - p_B}\right)^{\gamma+\delta+3} \Gamma\left[\gamma + \delta + 3, \frac{(p - p_B) V_0}{T}\right]$$

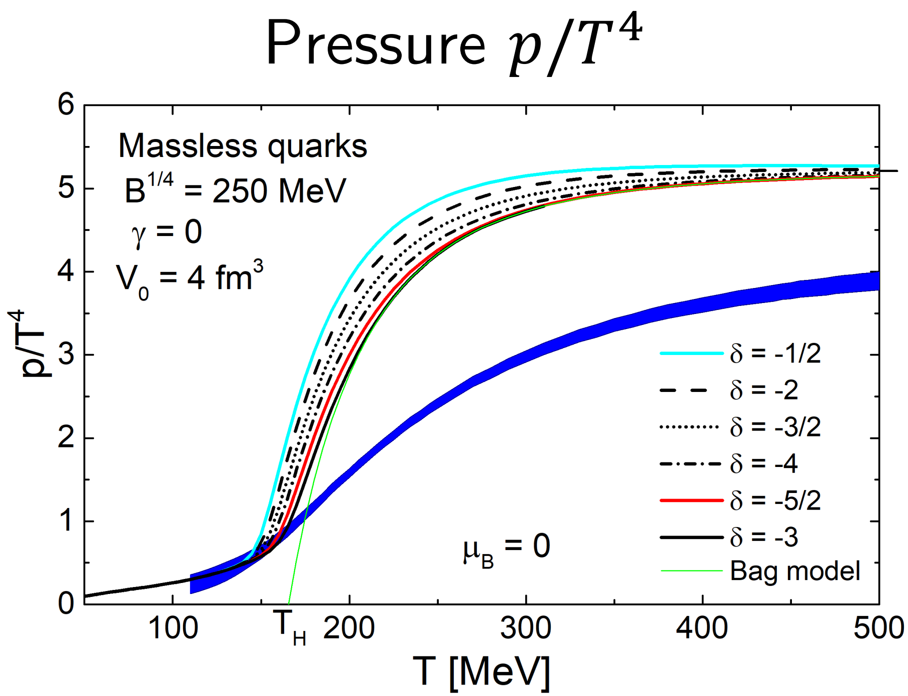
Solved numerically

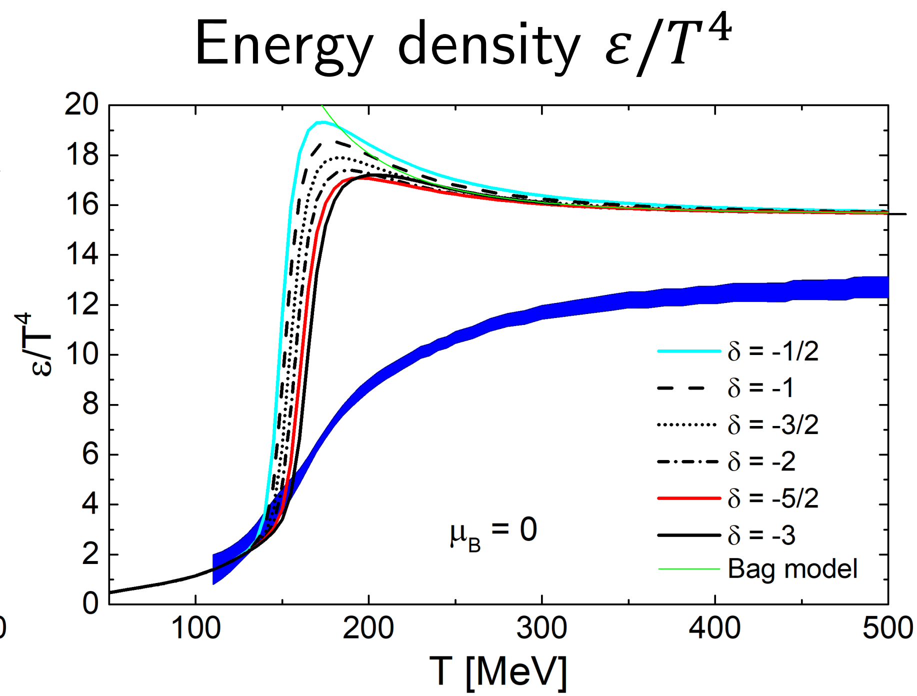
Calculation setup:

$$\gamma = 0, \quad -3 \leq \delta \leq -\frac{1}{2}, \quad B^{1/4} = 250 \text{ MeV}, \quad C = 0.03 \text{ GeV}^{-\delta+2}, \quad V_0 = 4 \text{ fm}^3$$

$$T_H = \left(\frac{3B}{\sigma_Q}\right)^{1/4} \simeq 165 \text{ MeV}$$

Thermodynamic functions



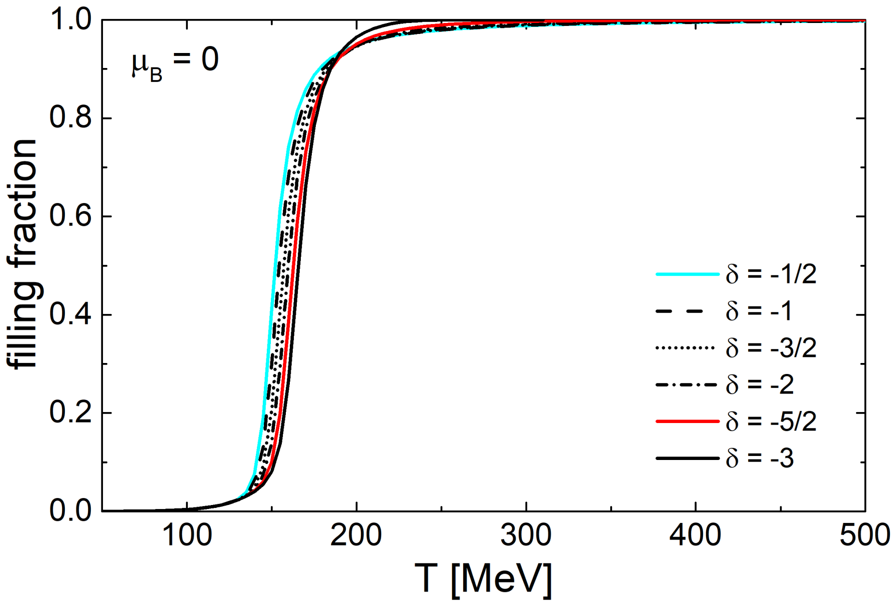


Lattice data from 1309.5258 (Wuppertal-Budapest)

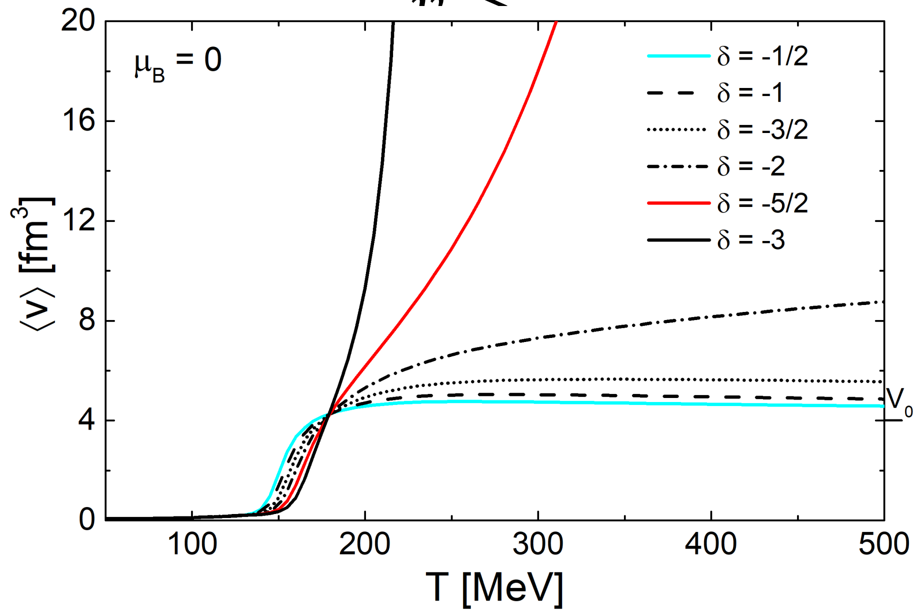
- Crossover transition towards bag model EoS
- Dependence on δ is mild
- Approach to the Stefan-Boltzmann limit is too fast
- Peak in energy density, not seen on the lattice

Nature of the transition

Filling fraction $= \frac{<V_{had}>}{V}$



Mean hadron volume $< v >$

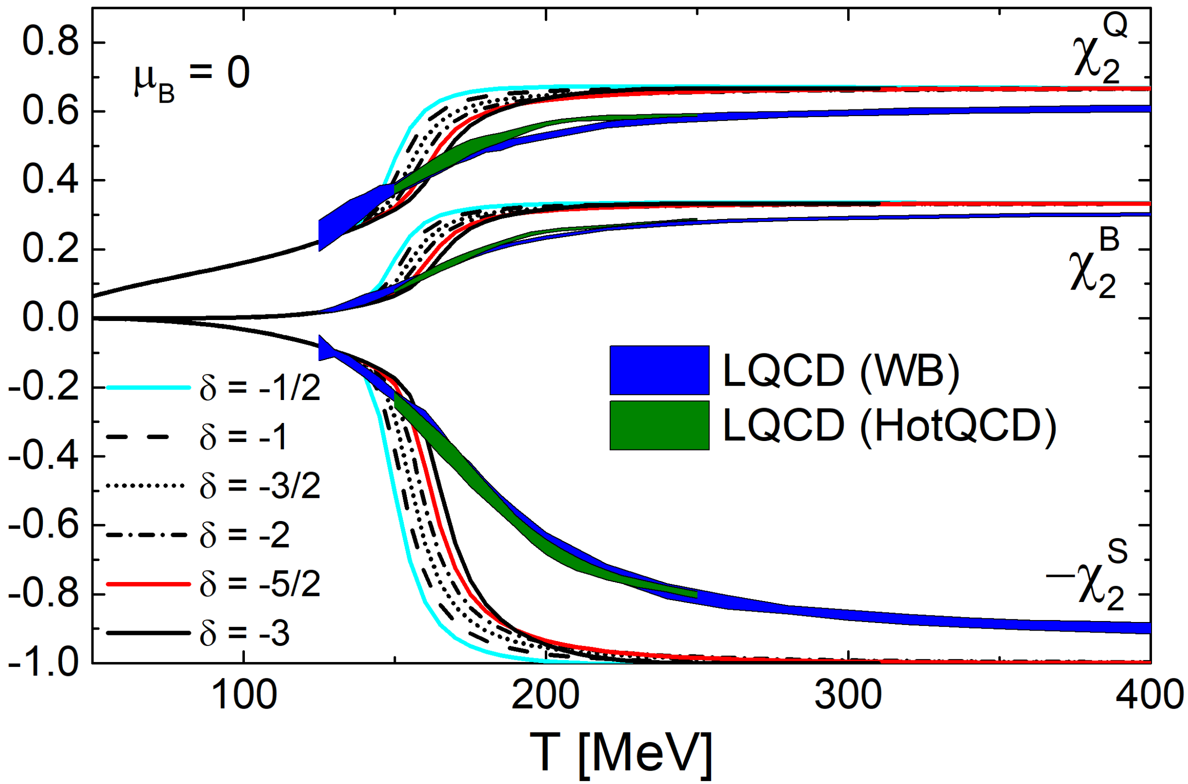


- Bags occupy almost whole space at large temperatures
- Strongest changes take place in the vicinity of T_H
- At $\delta < -7/4$ and $T \to \infty$ whole space — large bags with QGP

Conserved charges susceptibilities

$$\chi^{BSQ}_{lmn} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l\,\partial(\mu_S/T)^m\,\partial(\mu_Q/T)^n}$$

Available from lattice QCD, not considered in this type of model before



Qualitatively compatible with lattice QCD

Bag model with massive quarks

Main source of quantitative disagreement comes from inaccuracy of the standard MIT bag model with massless quarks for describing QGP

Quasiparticle models suggest sizable thermal masses of quarks and gluons in high-temperature QGP[Peshier et al., PLB '94; PRC '00; PRC '02]

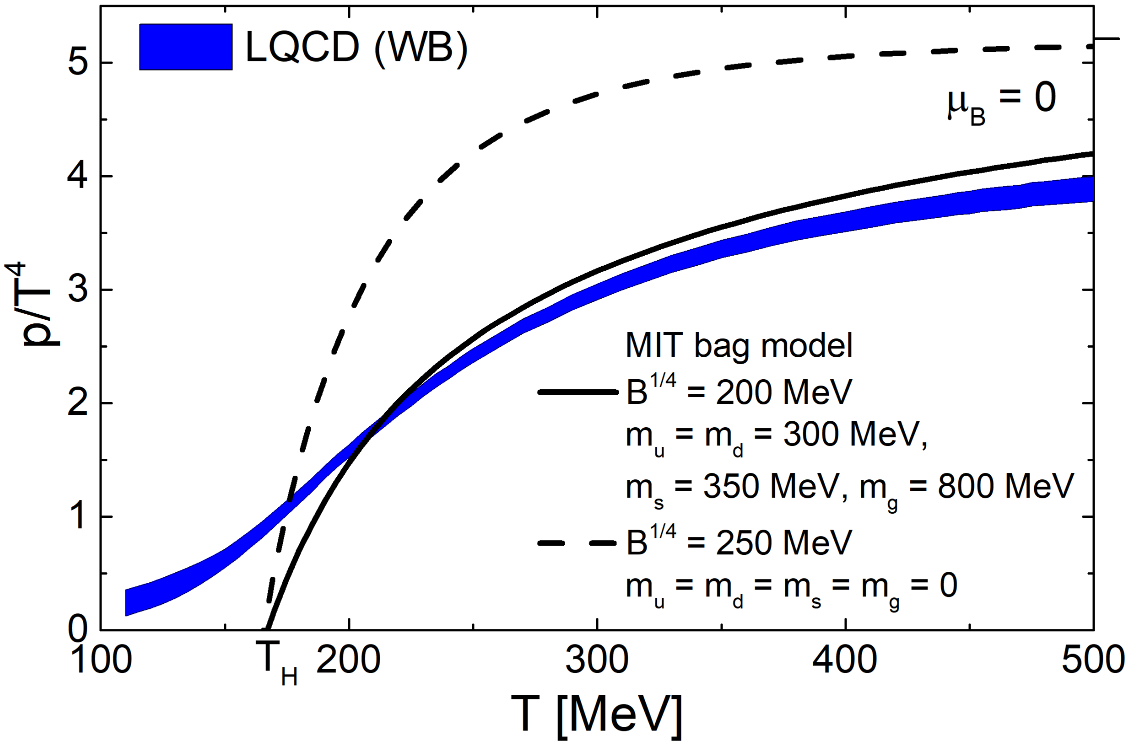
Heavy-bag model: bag model EoS with non-interacting *massive* quarks and gluons and the bag constant[Ivanov et al., PRC 72, 025804 (2005)]

Massive quarks and gluons instead of massless ones:

$$\sigma_Q(T,\lambda_B,\lambda_Q,\lambda_S) = \frac{8}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_g^2}}\left[\exp\left(\frac{\sqrt{k^2+m_g^2}}{T}\right)-1\right]^{-1}$$
$$+\sum_{f=u,d,s}\frac{3}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_f^2}}\left[\lambda_f^{-1}\exp\left(\frac{\sqrt{k^2+m_f^2}}{T}\right)+1\right]^{-1}$$
$$+\sum_{f=u,d,s}\frac{3}{\pi^2\,T^4}\int_0^\infty dk \frac{k^4}{\sqrt{k^2+m_f^2}}\left[\lambda_f\exp\left(\frac{\sqrt{k^2+m_f^2}}{T}\right)+1\right]^{-1}$$

Bag model with massive quarks

Introduction of constituent masses leads to much better description of QGP



Parameters for the crossover model:

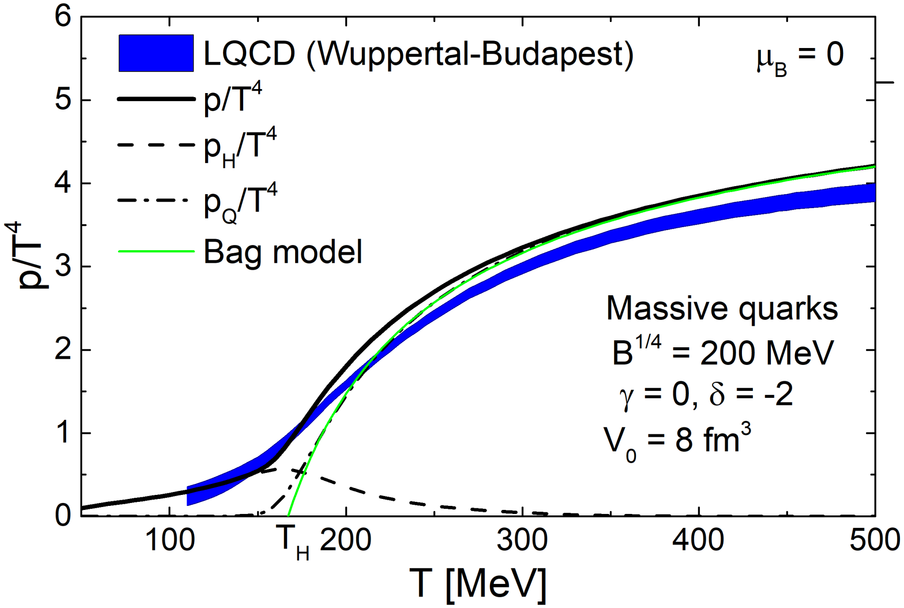
$$m_u = m_d = 300 \text{ MeV}, \quad m_s = 350 \text{ MeV}, \quad m_g = 800 \text{ MeV}, \quad B^{1/4} = 200 \text{ MeV}$$

$$\gamma = 0, \quad \delta = -2, \quad C = 0.03, \quad V_0 = 8 \text{ fm}^3$$

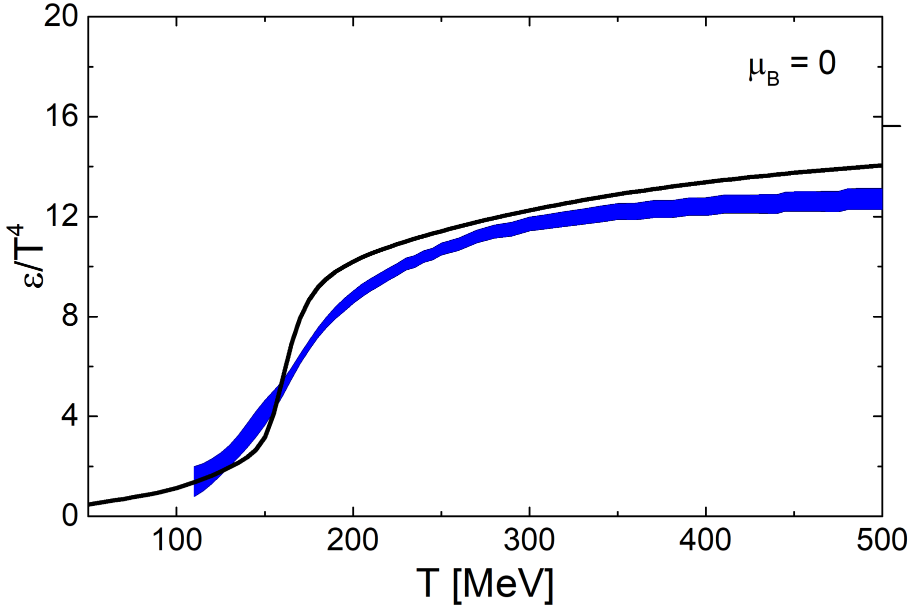
$T_H \simeq 167$ MeV

Hagedorn model: Thermodynamic functions

Pressure p/T^4



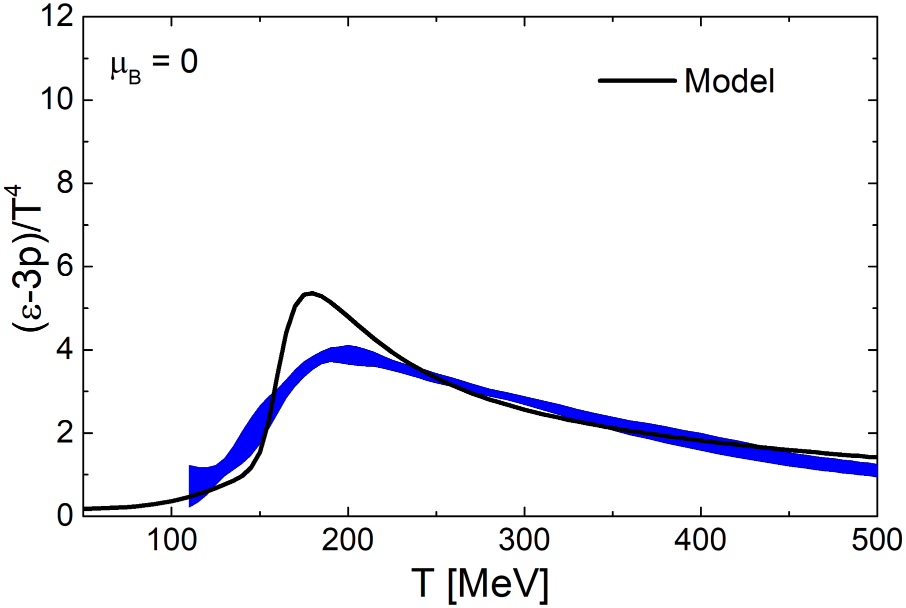
Energy density ε/T^4



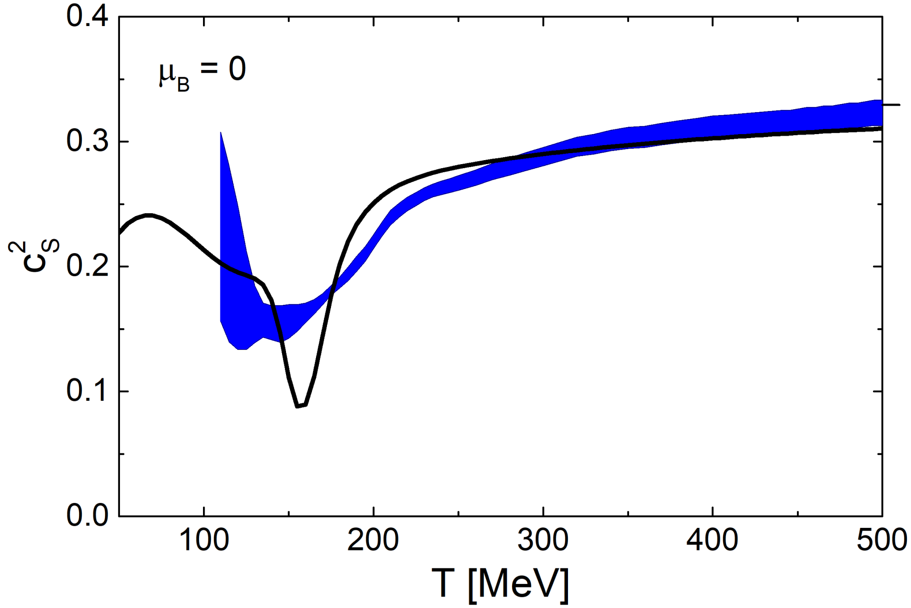
- Semi-quantitative description of lattice data
- Peak in energy density gone!

Hagedorn model: Thermodynamic functions

Trace anomaly $(\varepsilon - 3p)/T^4$

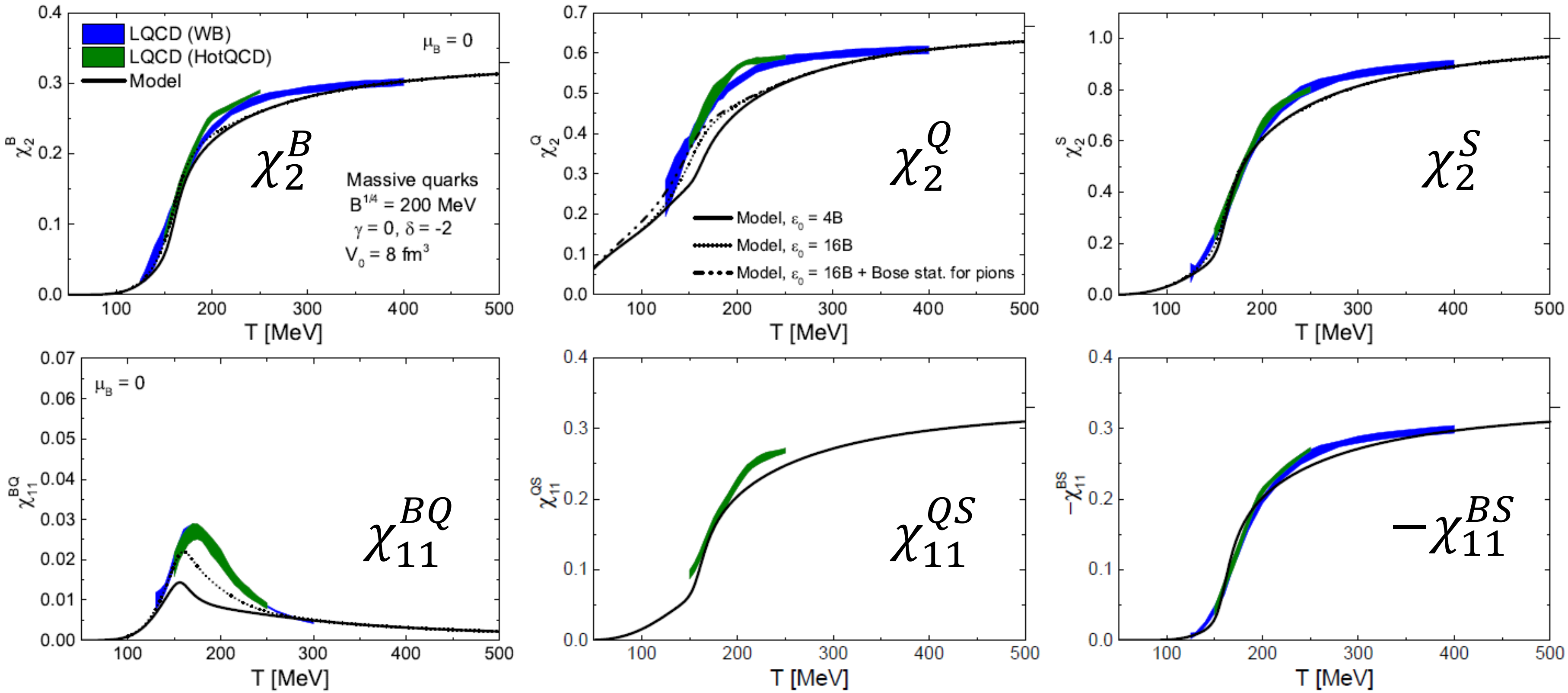


Speed of sound $c_s^2 = dp/d\varepsilon$



Hagedorn model: Susceptibilities

$$\chi^{BSQ}_{lmn} = \frac{\partial^{l+m+n} p/T^4}{\partial(\mu_B/T)^l\,\partial(\mu_S/T)^m\,\partial(\mu_Q/T)^n}$$



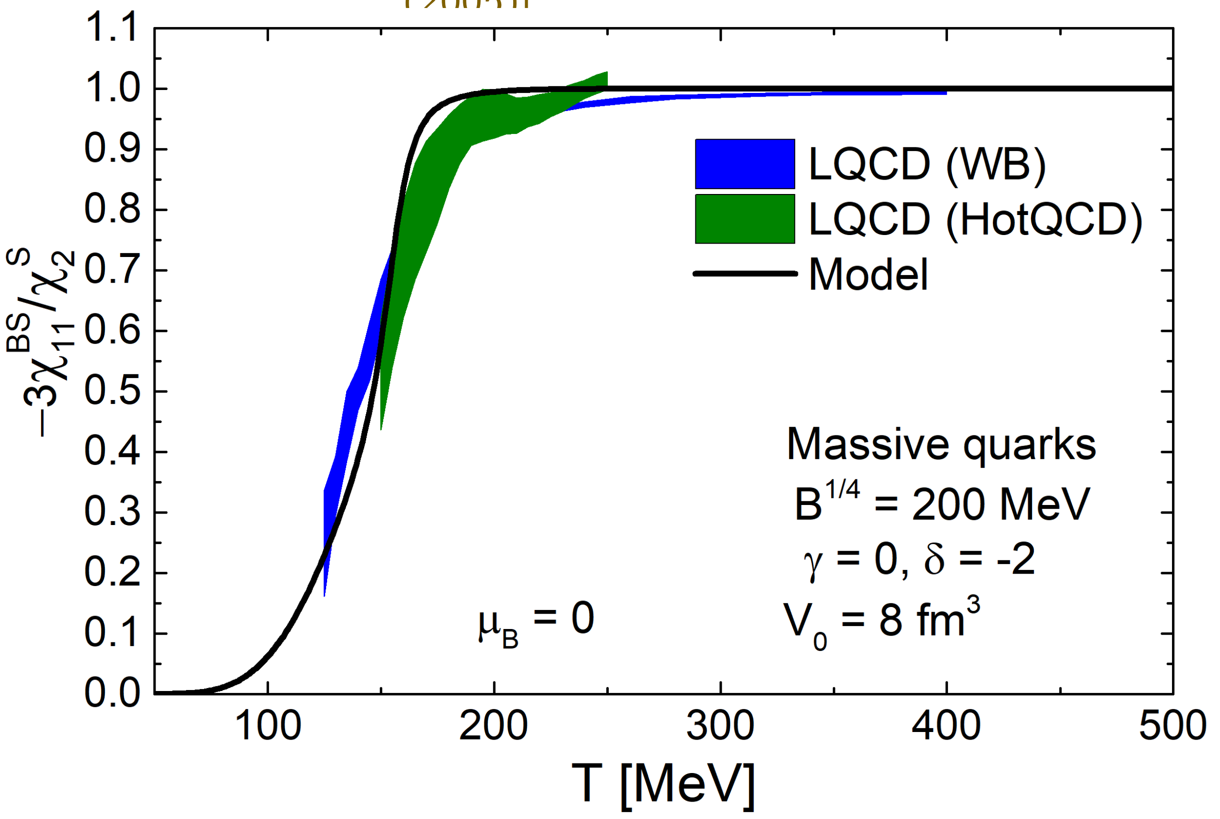
Lattice data from 1112.4416 (Wuppertal-Budapest), 1203.0784 (HotQCD)

Hagedorn model: Baryon-strangeness ratio

$$C_{BS} = -\frac{3\chi_{11}^{BS}}{\chi_2^S}$$

Useful diagnostic of QCD matter

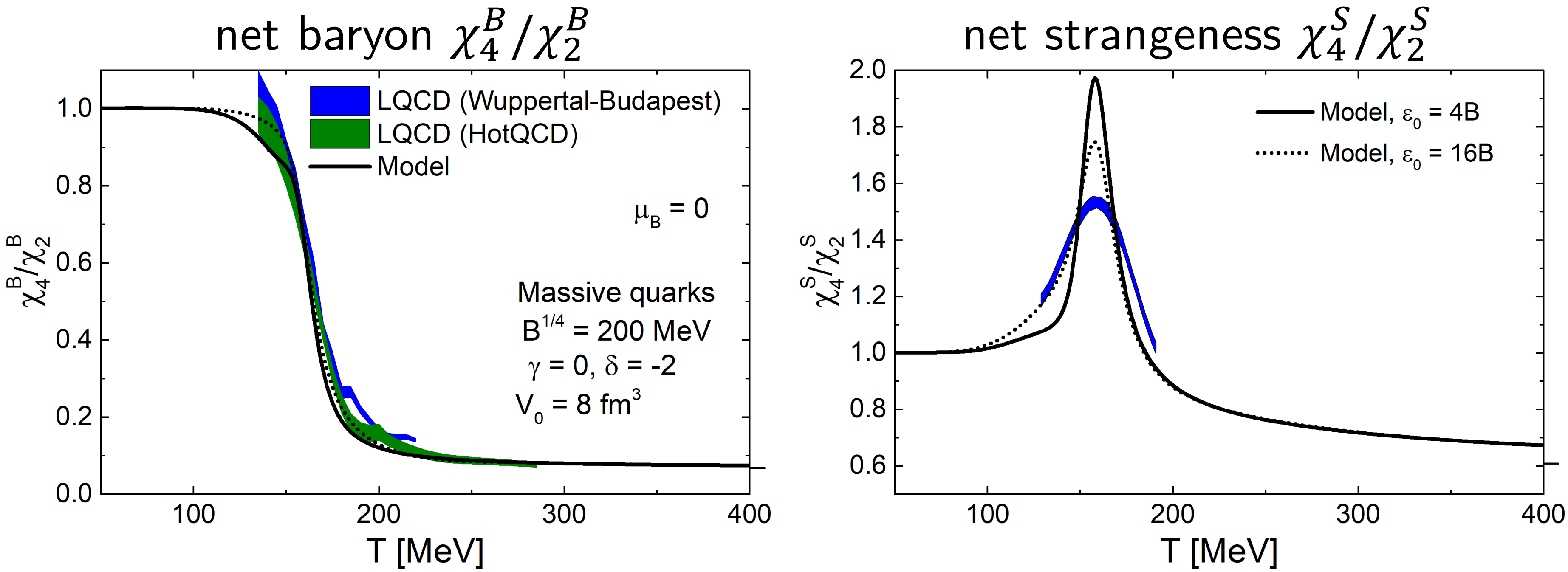
[V. Koch, Majumder, Randrup, PRL 95, 182301 (2005)]



Well consistent with lattice QCD

Hagedorn model: Higher-order susceptibilities

Higher-order susceptibilities are particularly sensitive probes of the parton-hadron transition and possible remnants of criticality at $\mu_B = 0$



Lattice data from 1305.6297 & 1805.04445 (Wuppertal-Budapest), 1708.04897 (HotQCD)

- Drop of χ_4^B/χ_2^B caused by repulsive interactions which ensure crossover transition to QGP
- Peak in χ_4^S/χ_2^S is an interplay of the presence of multi-strange hyperons and repulsive interactions